# Discrete-Time Minimum Tracking Based on Stochastic Approximation Algorithm With Randomized Differences

Oleg Granichin, Lev Gurevich, Alexander Vakhitov

***Abstract*— In this paper application of the stochastic approximation algorithm with randomized differences to the minimum tracking problem for the non-constrained optimization is considered. The upper bound of mean-squared estimation error is derived in the case of once differentiable functional and almost arbitrary observation noise. Numerical simulation of the estimates stabilization for the multidimensional optimization with unknown but bounded deterministic noise is provided. Stabilization bound has sufficiently small level comparing to significant level of noise.**

## I. INTRODUCTION

Stochastic approximation was introduced in a 1951 article in the *Annals of Mathematical Statistics* by Robbins and Monro [1] and was further developed for optimization problems by Kiefer and Wolfowitz [2]. It originally appeared as a tool for statistical computations, it was further developed to the separate field of control theory. Now this topic has wide variety of applications in such areas as adaptive signal processing, adaptive resource allocation in communication networks, system identification, adaptive control, etc.

The applications of stochastic approximation algorithms arise in the field of adaptive systems. SA algorithms have properties allowing to track typical behavior of a system in uncertain environment. The algorithm analyzed in this article is also very computationally efficient and needs very small amount of memory. It makes the algorithm applicable in highly dynamic environment.

The properties mentioned above make the SA algorithms applicable in such a new field as soft computing. They are used there for "parameter tuning." Notable among these are algorithms for training neural networks and algorithms for reinforcement learning, a popular learning paradigm for autonomous software agents with applications in e-commerce, robotics, etc. They are also widely applied in economic theory, providing a good model for collective phenomena, when the algorithm models behavior of individual bounded rational agents.

Non-stationary optimization problems can be described in discrete or continuous time. In our paper we consider only discrete time models. Let $f(x, n)$ be a functional we are optimizing at the moment of time $n$ ($n \in \mathbf{N}$). In [3] the Newton method and gradient method are applied to problems like that, but they are applicable only in case of two times differentiable functionals and $l < \nabla^2 f(x, n) < L$. Both methods require the possibility of direct gradient measurement in arbitrary point.

Algorithms of the SPSA-type with one or two measurements per each iteration appeared in papers of different researchers in the end of the 1980s [4], [7], [5], [6]. These algorithms are known for their applicability to problems with almost arbitrary noise [8], [9]. Moreover, the number of measurements made on each iteration is only one or two and is independent from the number of dimensions $d$ of the state space. This property sufficiently increases the number of function evaluations needed for the algorithms iteration in multidimensional case ($d >> 1$) while the asymptotic rate of convergence is not changed. Detailed review of development of such methods is provided in [8], [10].

Stochastic approximation algorithms were initially proven in case of the stationary functional. The gradient algorithm for the case of minimum tracking is provided in [3], however the stochastic setting is not discussed there. Further development of these ideas could be found in paper [11], where conditions of drift pace were relaxed. The book [12] uses the ordinary differential equations (ODE) approach to describe stochastic approximation. It addresses the issue of applications of stochastic approximation to tracking and time-varying systems in a following way: it is proven there that when the step size goes to zero in the same time as the number of the algorithm's iterates over a finite time interval tends to infinity, then the minimum estimates tend to true minimum values. This is not the case here, since we consider the number of iterates per unit of time to be fixed. In this paper we consider an application of simultaneous perturbation stochastic approximation algorithm to the problem of tracking of the functional minimum. SPSA algorithm does not rely on direct gradient measurement and is more robust to non-random noise than gradient-based methods mentioned earlier. The closest case was studied in [13], but we do not use the ODE approach and we establish more wide conditions for the estimates stabilization. In the following section we will give a problem statement that is more general than in [3], [11], in the third section we provide the algorithm, in the fourth section stabilization of the algorithm's estimates in the sense of the existence of the asymptotic upper bound for the mean squared estimation error is proved, then in the last section a numerical example illustrating stabilization is presented.

O. Granichin, L. Gurevich and A. Vakhitov are with Department of Mathematics and Mechanics, Saint Petersburg State University,198504 Saint Petersburg, Russia oleg_granichin@mail.ru, gurevich.lev@gmail.com, av38@yandex.ru

## II. PROBLEM STATEMENT

Consider the problem of minimum tracking for an averaged risk functional:

$$f(x,n)=E_w\{F(x,w,n)\}\to\min_x, \tag{1}$$

where $d,p\in\mathbf{N}$, $x\in\mathbf{R}^d$, $w\in\mathbf{R}^p$, $n\in\mathbf{N}$, $w$ is defined on the basic probability space $\{\Omega,\mathcal{F},P\}$, $E_w\{\cdot\}$ — mean value conditioned on the minimal $\sigma$-algebra in which $w$ is measurable.

*The goal* is to estimate $\theta_n$ — minimum point of functional $f(x,n)$, changing over time: $\theta_n=argmin_x f(x,n)$.

Let us assume that on the iteration we can measure:

$$y_n=F(x_n,w_n,n)+v_n, \tag{2}$$

where $x_n$ is an arbitrary measurement point chosen by algorithm, $w_n$ is a random value, that represents non-controlled uncertainty and $v_n$ is an observation noise.

Time in our model is discrete and implemented in number of iteration $n$.

## III. ALGORITHM

In this section we introduce a modification of SPSA algorithm provided by H.-F. Chen et al [14], which takes one perturbed and one non-perturbed measurement on each step. In this paper we try to illustrate the possibility of using such an algorithm for minimum tracking both theoretically by proving its convergence when certain conditions are satisfied and practically by providing a numerical example.

Let the perturbation sequence $\{\Delta_n\}$ be an independent sequence of Bernoulli random vectors, with component values $\pm1/\sqrt{d}$ with probability $\frac{1}{2}$. Let vector $\hat\theta_0\in\mathbb{R}^d$ be the initial estimation. We will estimate a sequence of minimum points $\{\theta_n\}$ with sequence $\{\hat\theta_n\}$ which is generated by the algorithm with fixed stepsize $\alpha\in\mathbf{R}$ and parameter $\beta\in\mathbf{R}$ $\alpha,\beta>0$, which is applied to the observations model (2) :

$$\begin{cases} x_{2n-1}=\hat\theta_{2n-2},\ x_{2n}=\hat\theta_{2n-2}+\beta\Delta_n,\\ \\ \hat\theta_{2n}=\hat\theta_{2n-2}-\frac{\alpha}{\beta}\Delta_n(y_{2n}-y_{2n-1}), \hat\theta_{2n-1}=\hat\theta_{2n-2}. \end{cases} \tag{3}$$

## IV. STABILIZATION OF ESTIMATES

To analyze the quality of estimates we will apply the following definition:

**Definition** A sequence of mean-squared estimation errors $E\|\hat\theta_n-\theta_n\|^2$ has *upper bound* $L>0$ if

$$\overline{\lim_{n\to\infty}}(E\|\hat\theta_n-\theta_n\|^2)^{1/2}\le L.$$

This definition is similar to the definition of $L_p$-stability in the work of Guo [15] ($p=2$).

Further we will assume that the following conditions are true.

**(A)** Function $f(\cdot,n)$ is strictly convex for each $n$ and some $\mu\in\mathbf{R}$, $\mu>0$: $\langle\nabla f(x,n),x-\theta_n\rangle\ge\mu\|x-\theta_n\|^2$.

**(B)** Gradient $\nabla F(\cdot,w,n)$ is Lipschitz $\forall n,\forall w$ with parameter $M\in\mathbf{R}$, $M>0$:
$\|\nabla F(x,w,n)-\nabla F(y,w,n)\|\le M\|x-y\|$.

**(C)** Local Lebesgue property for the function $\nabla F(x,w,n)$: $\forall x\in\mathbb{R}^d\,\exists$ neigbourhood $U_x$ such that $\forall x'\in U_x$ $\|\nabla F(x,w,n)\|<\Phi_{x,n}(w)$ where $\Phi_{x,n}(w):\mathbb{R}^p\to\mathbb{R}$ is integrable by $w$: $\int_{\mathbb{R}^p}\Phi_x(w)dw<\infty$

**(D)** Boundedness of the gradient of $F(x,w,n)$ in the minimum point, with $F_1,F_2\in\mathbf{R}$, $F_1>0$, $F_2>0$: $E\|\nabla F(\theta_n,w_n,n)\|\le F_1$, $E\|\nabla F(\theta_n,w_n,n)\|^2\le F_2$;

Let us denote expectation conditioned on random values $\theta_1\ldots,\theta_n$, $\hat\theta_1,\ldots,\hat\theta_n$ as $E_n\{\cdot\}$, conditioned on $\theta_1\ldots,\theta_n,\theta_{n+1},\theta_{n+2}$, $\hat\theta_1,\ldots,\hat\theta_n$ as $\bar{E}_n\{\cdot\}$.

Drift satisfies:

**(E)** In case of random drift,

$$\theta_{n-1}-E_{n-1}\theta_n\le A_0,$$

$$E_{n-1}\|\theta_n-\theta_{n-1}\|\le A_1,$$

$$E_{n-1}\|\theta_n-\theta_{n-1}\|^2\le A_2,$$

$$E_{n-2}(\theta_n-\theta_{n-1})^T(\theta_{n-1}-\theta_{n-2})\le A_3.$$

If drift is not random, then

$$\theta_{n-1}-E_{n-1}\theta_n\le A_0,$$

$$\|\theta_n-\theta_{n-1}\|\le A_1=A_2=A_3.$$

**(F)** Constraint on the function values' change in arbitrary point $x$ with $C_i^j\in\mathbf{R}$ for $i=1,2$, $j=0,1,2$:

$$E_n\{F(x,w_n,n)-F(x,w_{n-1},n-1)\}\le C_1^1\|x-\theta_{n-1}\|+C_1^0,$$

$$E_n(F(x,w_n,n)-F(x,w_{n-1},n-1))^2\le$$

$$C_2^2\|x-\theta_{n-1}\|^2+C_2^1\|x-\theta_{n-1}\|+C_2^0.$$

**(G)** The observation noise $v_n$ satisfies: $|v_{2n}-v_{2n-1}|\le\sigma_1$, or if it has statistical nature then: $E_{2n-1}(v_{2n}-v_{2n-1})^2\le\sigma_2$, $E_{2n-1}|v_{2n}-v_{2n-1}|\le\sigma_1$.

Here we should make several notes:

1). Sequence $\{v_n\}$ could be of non-statistical but unknown deterministic nature. 2). In the typical case of $f(x,n)=F(x,w,n)$ both constants $F_1,F_2$ from the condition **(D)** are equal to 0. 3). Constraint **(E)** allows both random and deterministic drift. Similar condition is introduced in [3], it is slightly relaxed in [11]. In [9] the following drift model is proposed:

$$(\mathbf{E}')\quad \theta_n=D_1\theta_{n-1}+D_2+\xi_n,$$

where $\xi_n$ is random value. If the matrix $A^1$ is known it is easy to include it directly to the algorithm. For $D_1=I$ (unit diagonal matrix) condition **(E)** is more general than **(E')** with $A_0=-D_2-E\xi_n$, $A_1=\|D_2\|+E\|\xi_n\|$, $A_2=\|D_2\|^2+2\|D_2\|E\|\xi_n\|+E\|\xi_n\|^2$.

In this paper we will only consider drift constraints in the form **(E)**. Existence of finite asymptotically effective bound for the estimates under the condition **(E)** implies the applicability of the algorithm proposed to a wide variety of problems.

**(H)** We will further assume that random values $\Delta_n$ are not dependent on $\theta_k$, $w_k$ , $\hat\theta_0$ and on $v_k$ (if they are assumed to have random nature) for $k = 1, 2, \ldots, 2n$ and random sequences $\{\theta_k\}$ and $\{w_k\}$ are independent.

Let us define $K_2 = M^2 + 2\frac{C_1^1 M}{\beta} + \frac{C_2^2}{\beta^2}$, $k = \alpha^2 K_2 - 2\frac{\alpha\mu}{d} + 1$, $H_2 = \beta M^2 + 2MF_1 + C_1^1 M + 2\frac{M(\sigma_1 + C_1^0) + C_1 F_1}{\beta} + \frac{2\sigma_1 C_1^1 + C_2^1}{\beta^2}$, $h = \alpha^2(H_2 + 4A_0 K_2) + \alpha(\beta M - 8\frac{\mu A_0}{d}) + 4A_0$, $L_2 = \beta^2 \frac{M^2}{4} + \beta F_1 M + F_2 + C_1^0 M + \sigma_1 M + 2\frac{C_1^0 F_1 + \sigma_1 F_1}{\beta} + \frac{C_2^0 + \sigma_2 + 2\sigma_1 C_1^0}{\beta^2}$, $l = \alpha^2(L_2 + 2A_1 H_2 + 2(A_2 + A_3)K_2) + \alpha(2\beta A_1 M - 4(A_2 + A_3)\frac{\mu}{d}) + 2(A_2 + A_3)$.

**Theorem 4.1**. Assume that conditions **(A)-(H)** on functions $f$, $F$ and $\nabla F$ and values $\theta_n$, $\hat{\theta}_n$, $v_n$, $w_n$, $y_n$ and $\Delta_n$ are satisfied. The constants $\alpha, \beta > 0$ are chosen as $0 < k < 1$.

*Then* the sequence of mean-squared estimation errors provided by the algorithm (3) have upper bound $L$: $\overline{\lim}_{n\to\infty} (E\|\hat{\theta}_n - \theta_n\|^2)^{1/2} \le L$,

$$L = \frac{h}{2(1-k)}\left(1 + \sqrt{1 + \frac{4l(1-k)}{h^2}}\right),$$

and they can be bounded as

$$E\|\hat{\theta}_n - \theta_n\|^2 \le (k + \epsilon/2)^n \|\hat{\theta}_0 - \theta_0\|^2 + (1 - (k+\epsilon/2)^n)L^2,$$

where $\epsilon = \frac{h^2}{2l}(\sqrt{1 + 4l(1-k)h^{-2}} - 1)$. Furthermore, the following property holds:

$$P\{\forall n\ \forall \lambda > 0\ \|\hat{\theta}_n - \theta_n\|^2 \le L^2 + (k+\epsilon/2)^n(\|\hat{\theta}_0 - \theta_0\|^2 + \lambda)\} \ge$$
$$\ge 1 - \frac{E\|\hat{\theta}_0 - \theta_0\|^2}{\lambda}.$$

See the proof of Theorem 4.1 in the appendix. Conditions **(A)–(D),(G)–(H)** are standard for SPSA algorithms [8]. Conditions **(E),(E'), (F)** are related to drift. See the proof of Theorem 4.1 in appendix.

The next result is inspired by the work of L. Guo and L. Ljung [17] where the performance of linear tracking depending on the step-size parameter is described. Least mean squares, recursive least squares and Kalman-type algorithms are chosen for analysis. The authors demonstrate a trade-off in performance of the algorithms between the sensitivity of the algorithm estimates to noise and the size of the drift. In this paper, we consider a nonlinear case.

**Theorem 4.2**. Assume the conditions of the Theorem 4.1 hold.

*Then* following asymptotic expansion holds:

$$L = \frac{2A_0}{\mu\alpha} + 2(\frac{\beta M}{4\mu} - 2A_0) - \frac{A_0 K_2}{\mu^2} + \frac{1}{4A_0} + \frac{A_2 + A_3}{4A_0\mu} +$$
$$+ \alpha\big((32A_0^2\mu)^{-1}\beta M(\mu^2 + A_2 + A_3 - 1) +$$
$$+ (16\mu^2 A_0)^{-1}(4\mu^3 + 4\mu^2(1 - A_2 - A_3) + K_2\mu - K_2(A_2 + A_3)) -$$
$$- \frac{K_2(\beta M + 2A_0)}{2\mu^2} + \frac{H_2 + 4A_0 K_2}{\mu}\big) + o(\alpha)\ (\alpha \to 0),$$

where $H_2 = 2MF_1 + C_1 M + \frac{\beta M}{2} + 2\frac{M(\sigma_1 + D_1) + C_1 F_1}{\beta} + \frac{2\sigma_1 C_1 + C_2^2}{\beta^2}$ and $K_2 = M^2 + 2\frac{C_1 M}{\beta} + \frac{C_2^1}{\beta^2}$.

The result of the Theorem 2 shows that to make an effect of drift smaller, $\alpha$ should be made bigger (see the first term).

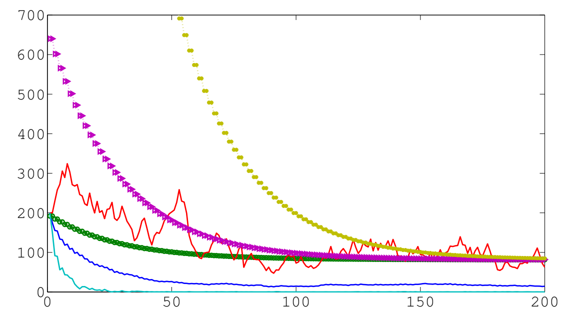


Fig. 1. Case of 2-dimensional drift for $0.5\|x - \theta_n\|^2 + v_n,\ x, \theta_n \in \mathbf{R}^2$. There were 100 runs of the algoritms performed, each with 100 iterations starting from the point $\hat{\theta}_0 = (10.0, 10.0)^T$. The solid lines are minimal, average and maximal squared estimation error $\|\hat{\theta}_n - \theta_n\|^2$.. Line marked with circles is the theoretical bound according to the first statement of the Theorem 4.1, line with triangles is the probabilistic 50%-boundary and line marked with stars is 95-% boundary, according to the second statement of the Theorem 4.1.

Also, noise level is a constituent of $H_2$ which is multiplied by $\alpha$, making effect of noise arbitrarily small with small $\alpha$. So, there is a tradeoff between minimizing the effects of noise and sustaining the drift. This tradeoff was demonstrated also in case of the linear system [9] for slightly different drift model.

The formula presented in the Theorem 4.2 can be used for optimal choice of step size (sometimes called gain) $\alpha$ when $\alpha$ is supposed to be small.

Earlier the proof of the Theorem similar to 4.1 was given in [16] with more strict conditions. Here the bound is improved so that Theorem 4.2 holds.

## V. EXAMPLES

Simple practical application of the algorithm (3) is the estimation of the multidimensional moving point coordinates when only information about distance from arbitrary point to the moving point is available with additive noise. As a result of Theorem 4.1, the algorithm (3) provides the point estimate in case of limited drift of the point and limited observation noise. In [16] the drift with formula $\theta_n = \theta_{n-1} + \zeta$ was considered, where $\zeta$ is uniformly distributed on the sphere: $\|\zeta\| = 1$. Here we also provide this example and show how theoretical results about the behavior of the algorithm optimizing this function correspond to the practical evidence. Firstly we consider the case when the dimension of argument $d = 2$.

The function $f(x, n) = F(x, w, n) = \frac{1}{2}\|x - \theta_n\|^2$ was measured with additional non-random unknown but bounded noise sequence $\|v_n\| \le 1$. This sequence was generated using formulas $v_{2n} = 1 - (n \bmod 3)$ and $v_{2n-1} = 1 - (n \bmod 7)/3$.

In this case we have $\mu = M = 1$, $F_1 = F_2 = 0$, $A_0 = 0$, $A_1 = A_2 = A_3 = 1$, $C_1^1 = C_2^2 = 1$,$C_1^0 = -0.5$, $C_2^1 = -1$, $C_2^0 = 0.25$. $\sigma_1 = 2$, $\sigma_2 = 4$. The estimates have shown convergence to the theoretically proven interval. This example is illustrated at Figure 1.

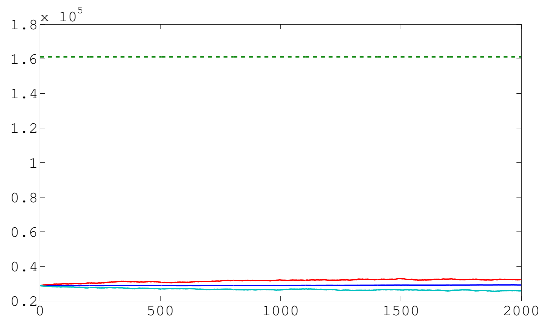


Fig. 2. Case of 100-dimensional drift for $0.5\|x-\theta_n\|^2+v_n,\ x,\theta_n\in\mathbf{R}^{100}$. There were 100 runs of the algoritms performed, each with 1000 iterations starting from the point $\hat{\theta}_0=(17.0,17.0,...,17.0)^T$. The solid lines are minimal, average and maximal squared estimation error $\|\hat{\theta}_n-\theta_n\|^2$. Dashed line is the theoretical bound $L^2$.

The next example is with the same function and noise as before, but the dimension of argument is equal to 100. This example is aimed to demonstrate good behavior of the algorithm's estimates in highly dimensional spaces, see Figure 2. The theoretical boundary is far from precise.

## VI. CONCLUSION

In our work we apply the SPSA-type algorithm to the problem of extreme point tracking with almost arbitrary noise. Drift is assumed to be limited, which includes random and directed drift. It was proven that the estimation error of this algorithm is limited with constant value. The modeling was performed on a multidimensional case.

The authors want next to prove more precise boundaries of the estimation error. The stabilization of estimates for arbitrary $p$ rather than for $p=2$ (as in this paper) could be considered. It could be also interesting to modify the algorithm to work with unknown polynomial drift, using the technique of polynomial fitting demonstrated in [19].


## VII. ACKNOWLEDGEMENTS

We would like to thank professors B.T. Polyak and V. Ya. Katkovnik as well as anonymous reviewers for their valuable comments.


## VIII. APPENDIX

*Lemma 1.* If $v_n^2\le kv_{n-1}^2+hv_{n-1}+l$ $n\in\mathbf{N}, k\in(0,1)$. Then $v_n^2\le(k+\epsilon/2)^n v_0^2+(\frac{h}{2(1-k)}(1+\sqrt{1+4l(1-k)h^{-2}}))^2$, or $\overline{\lim}_{n\to\infty}v_n\le\frac{h}{2(1-k)}(1+\sqrt{1+4l(1-k)h^{-2}})$, where $\epsilon=\frac{h^2}{2l}(\sqrt{1+4l(1-k)h^{-2}}-1)$.

*Proof.* $kv_{n-1}^2+hv_{n-1}+l\le(k+\epsilon/2)v_{n-1}^2+\frac{h^2}{2\epsilon}+l$, where $\epsilon>0$. For sufficiently small $\epsilon$, $k+\epsilon/2<1$, so

$$v_n\le(k+\epsilon/2)^n v_0+\frac{(\frac{h^2}{2\epsilon}+l)(1-(k+\epsilon/2)^n)}{1-k-\epsilon/2},$$

$$v_n\le(k+\epsilon/2)^n v_0+\frac{\frac{h^2}{2\epsilon}+l}{1-k-\epsilon/2}. \quad (4)$$

Minimizing the last term by $\epsilon$, we get $\epsilon_{min}=\frac{-h^2+\sqrt{h^4+4(1-k)lh^{-2}}}{2l}$. Obviously, $\epsilon_{min}>0$ because $\sqrt{h^2+4l(1-k)h^{-2}}>h$. Also, $\epsilon_{min}/2<1-k$. Substituting $\epsilon_{min}$ into the inequality, we get $\frac{\frac{h^2}{2\epsilon_{min}}+l}{1-k-\epsilon_{min}/2}=(\frac{h}{2(1-k)}(1+\sqrt{1+4l(1-k)h^{-2}}))^2$.

□

*Proof of Theorem 4.1.* According to (3) and the property $\|\Delta_n\|=1$ we have:

$$\|\hat{\theta}_{2n}-\theta_{2n}\|^2=\|\hat{\theta}_{2n-2}-\theta_{2n}\|^2-2\langle\hat{\theta}_{2n-2}-\theta_{2n},\frac{\alpha}{\beta}(y_{2n}-y_{2n-1})\Delta_n\rangle+\|\frac{\alpha}{\beta}(y_{2n}-y_{2n-1})\|^2. \quad (5)$$

Consider the second term in (5). We can say that

$$\bar{E}_{2n-2}\{\Delta_n(y_{2n}-y_{2n-1})\}=$$
$$=\bar{E}_{2n-2}\{\Delta_n F(\hat{\theta}_{2n-2}+\beta\Delta_n,w_{2n},2n)\}\le$$
$$\le\bar{E}_{2n-2}\{\Delta_n F(\hat{\theta}_{2n-2},w_{2n},2n)\}+$$
$$+\bar{E}_{2n-2}\{\Delta_n\langle\nabla F(\hat{\theta}_{2n-2},w_{2n},2n),\beta\Delta_n\rangle\}+\frac{M}{2}\beta^2. \quad (6)$$

The first term because of the property **(H)** is equal to 0, and the second term $\bar{E}_{2n-2}\{\Delta_n\langle\nabla F(\hat{\theta}_{2n-2},w_{2n},2n),\beta\Delta_n\rangle\}=\beta\frac{1}{d}\nabla f(\hat{\theta}_{2n},2n)$.

$$-\bar{E}_{2n-2}\{\langle\hat{\theta}_{2n-2}-\theta_{2n},\frac{\alpha}{\beta}\Delta_n(y_{2n}-y_{2n-1})\}\le$$
$$\le-\frac{\alpha\mu}{d}\|\hat{\theta}_{2n-2}-\theta_{2n}\|^2+\alpha\beta\frac{M}{2}\|\hat{\theta}_{2n-2}-\theta_{2n-2}\|.$$

Next, we analyze the third term in (5). We use the representation $y_{2n}-y_{2n-1}=F(\hat{\theta}_{2n-2}+\beta\Delta_n,w_{2n},2n)-F(\hat{\theta}_{2n-2},w_{2n},2n)+F(\hat{\theta}_{2n-2},w_{2n},2n)-F(\hat{\theta}_{2n-2},w_{2n-1},2n-1)$. Let us note that:

$$\bar{E}_{2n-2}\{F(\hat{\theta}_{2n-2}+\beta\Delta_n,w_{2n},2n)-F(\hat{\theta}_{2n-2},w_{2n},$$
$$2n)\}\le\beta M\|\hat{\theta}_{2n-2}-\theta_{2n}\|+\beta F_1+\beta^2\frac{M}{2};$$
$$\bar{E}_{2n-2}\{(F(\hat{\theta}_{2n-2}+\beta\Delta_n,w_{2n},2n)-F(\hat{\theta}_{2n-2},w_{2n},$$
$$2n))^2\}\le\beta^2M^2\|\hat{\theta}_{2n-2}-\theta_{2n}\|^2+2\beta M$$
$$(\beta F_1+\beta^2\frac{M}{2})\|\hat{\theta}_{2n-2}-\theta_{2n}\|+\beta^2F_2+\beta^3MF_1+\beta^4\frac{M^2}{4}.$$

Using the property **(F)** we get

$$\bar{E}_{2n-2}\{(\frac{\alpha}{\beta}\Delta_n(y_{2n}-y_{2n-1}))^2\}\le\|\hat{\theta}_{2n-2}-\theta_{2n}\|^2\frac{\alpha^2}{\beta^2}\cdot$$
$$\cdot(\beta^2M^2+C_2^2+2\beta MC_1^1)+\|\hat{\theta}_{2n-2}-\theta_{2n-1}\|\frac{\alpha^2}{\beta^2}(2\beta M(\beta F_1+$$
$$+\beta^2\frac{M}{2})+C_2^1+2C_1^1(\beta F_1+\beta^2\frac{M}{2})+2\beta MC_1^0+2\sigma_1C_1^1+$$
$$+2\sigma_1\beta M)+\frac{\alpha^2}{\beta^2}(\beta^2F_2+\beta^3F_1M+\beta^4\frac{M^2}{4}+C_2^0+\sigma_2+$$
$$+2(\beta F_1+\beta^2\frac{M}{2})C_1^0+2\sigma_1(\beta F_1+\beta^2\frac{M}{2})+2\sigma_1C_1^0). \quad (7)$$

We conclude that $\bar{E}_{2n-2}\{\|\hat{\theta}_{2n}-\theta_{2n}\|^2\} \leq k_1\|\hat{\theta}_{2n-2}-\theta_{2n}\|^2+h_1\|\hat{\theta}_{2n-2}-\theta_{2n}\|+l_1$, where $k_1=1-2\frac{\alpha\mu}{d}+\alpha^2(M^2+2\frac{C_1^1 M}{\beta}+\frac{C_2^2}{\beta^2})$, $h_1=\alpha^2(\beta M^2+2MF_1+C_1^1 M+2\frac{M(\sigma_1+C_1^0)+C_1^0 F_1}{\beta}+\frac{2\sigma_1 C_1^1+C_2^1}{\beta^2})+\alpha\beta M$, $l_1=\alpha^2(\beta^2\frac{M^2}{4}+\beta F_1 M+F_2+C_1^0 M+\sigma_1 M+2\frac{C_1^0 F_1+\sigma_1 F_1}{\beta}+\frac{C_2^0+\sigma_2+2\sigma_1 C_1^0}{\beta^2})$.

We need to consider two steps back, moving from $\theta_{2n}$ to $\theta_{2n-2}$ in the left side of inequality. We observe that $E_{2n-2}\{\|\hat{\theta}_{2n-2}-\theta_{2n}\|\} \leq \|\hat{\theta}_{2n-2}-\theta_{2n-2}\|+2A_1$; $E_{2n-2}\{\|\hat{\theta}_{2n-2}-\theta_{2n}\|^2\} \leq \|\hat{\theta}_{2n-2}-\theta_{2n-2}\|^2+4A_0\|\hat{\theta}_{2n-2}-\theta_{2n-2}\|+2A_2+2A_3$. To do those steps, we take the expectation $E_{2n-2}\{\cdot\}$ from the both sides of the inequality obtained.

$$E_{2n-2}\{\|\hat{\theta}_{2n}-\theta_{2n}\|^2\} \leq k\|\hat{\theta}_{2n-2}-\theta_{2n-2}\|^2+ \\ +h\|\hat{\theta}_{2n-2}-\theta_{2n-2}\|+l, \quad (8)$$

where $k=k_1$; $h=h_1+4A_0k_1$; $l=l_1+2A_1h_1+2(A_2+A_3)k_1$. By the conditions of the Theorem 4.1, $k\in(0,1)$. Applying the Lemma 1 to the sequence of unconditional expectations $E\{\|\hat{\theta}_{2n}-\theta_{2n}\|^2\}$ we finish the proof of the first statement.

The second statement is proved adapting the methodology of the Lemma 4 from [20]. Let us define $Z_n = \frac{\|\hat{\theta}_{2n}-\theta_{2n}\|^2-(k+\epsilon/2)^n E\|\hat{\theta}_0-\theta_0\|^2-L^2}{(k+\epsilon/2)^n}$. $Z_n$ is a super-martingale. Then according to [21] $P\{\max_{0\leq n\leq n_0} Z_n \geq \lambda\} \leq \lambda^{-1}(EZ_0+EZ_{n0}^-) \leq \frac{-L^2+L^2+(k+\epsilon/2)^n E\{\|\hat{\theta}_0-\theta_0\|^2\}}{\lambda(k+\epsilon/2)^n} \to_{n_0\to\infty} E\{\|\hat{\theta}_0-\theta_0\|^2\}\lambda^{-1}$, according to the first statement, where $Z_{n0}^- = -\min\{Z_{n0},0\}$. The result follows from $P\{\forall n \ Z_n \leq \lambda\} \geq 1-\frac{E\{\|\hat{\theta}_0-\theta_0\|^2\}}{\lambda}$. □

*Proof of Theorem 4.2* Let us get an expansion for $\frac{h}{1-k}$ while $\alpha\to 0$. Analyzing formula for $k$, we see that $\frac{h}{1-k}=\frac{1}{\alpha}\frac{a\alpha^2+b\alpha+c}{d+e\alpha}$ for $k=1-d\alpha-e\alpha^2$, $h=a\alpha^2+b\alpha+c$. We derive then $\frac{h}{1-k}\equiv\frac{c}{d\alpha}+\frac{bd-ce}{d^2}+\alpha(\frac{2(ad-be)-ce}{2d^2})+o(\alpha)$. That is, $\frac{h}{1-k}\equiv\frac{2A_0}{\mu\alpha}+\frac{\beta M}{2\mu}-4A_0-\frac{A_0K_2}{\mu^2}+\frac{2\mu H_2+(8A_0\mu-\beta M-2A_0)K_2}{4\mu^2}\alpha+o(\alpha)$.

Next, $\frac{l(1-k)}{h^2}\equiv\alpha\frac{a\alpha^3+b\alpha^2+c\alpha+d}{(f\alpha^2+g\alpha+p)^2}\equiv\frac{d}{p^2}\alpha+\frac{cp-2g}{p^3}\alpha^2+o(\alpha^2)$. Using the formula $\sqrt{1+\alpha(x)}\equiv 1+\frac{1}{2}\alpha(x)+o(\alpha(x))$ for $\alpha(x)\to 0$ $(x\to 0)$ we derive $1+\sqrt{1+l(1-k)h^{-2}}\equiv 2+\frac{\mu+A_2+A_3}{16A_0^2}\alpha+\frac{2A_0K_2+8A_0\mu-\beta M}{64A_0^3}\alpha^2+o(\alpha^2)$.

Finding the multiplication of both formulas derived before, we get the result expansion of the Theorem 4.2. □

## References


[1] Robbins H., Monro S., A Stochastic Approximation Method, *Annals of Mathematical Statistics* vol 22, No. 3, September 1951, pp. 400-407.

[2] Kiefer J, Wolfowitz J, Stochastic estimation of the maximum of a regression function, *Annals of Mathematical Statistics*, vol 23, No. 3, September 1952, pp. 462-466

[3] Polyak B. T., *Introduction to Optimization.* New York: Optimization Software; 1987.

[4] Granichin O. N., Procedure of stochastic approximation with disturbances at the input, *Automat. Remote Control*, vol. 53, No 2, part 1, 1992, pp. 232–237.

[5] Polyak B. T., Tsybakov A. B., Optimal order of accuracy for search algorithms in stochastic optimization, *Problems of Information Transmission*, vol. 26, No 2, 1990, pp. 126–133.

[6] Spall J. C. Multivariate stochastic approximation using a simultaneous perturbation gradient approximation. *IEEE Trans. Automat. Contr.*, vol. 37, 1992, pp. 332–341.

[7] Granichin O. N., A stochastic recursive procedure with dependent noises in the observation that uses sample perturbations in the input, *Vestnik Leningrad Univ. Math.*, vol. 22, No 1(4), 1989, pp. 27–31.

[8] Granichin O. N., Polyak B. T., *Randomized Algorithms of Estimation and Optimization Under Almost Arbitrary Noises.* Moscow: Nauka; 2003.

[9] Granichin O. N., Linear Regression and Filtering Under Nonstandard Assumptions (Arbitrary Noise). *IEEE Trans. Automat. Contr.*, vol. 49, No 10, 2004, pp. 1830–1837.

[10] Spall J. C., "Developments in stochastic optimization algorithms with gradient approximations based on function measurements" *Proc. of the Winter Simulation Conf.*,1994, pp. 207–214.

[11] Popkov A. Yu., Gradient methods for nonstationary unconstrained optimization problems. *Automat. Remote Control*, No. 6, 2005, pp. 883–891.

[12] Kushner H., Yin G., *Stochastic Approximation and Recursive Algorithms and Applications.* Springer; 2003.

[13] Borkar V. S. *Stochastic Approximation. A Dynamical Systems Viewpoint.* Cambridge University Press; 2008.

[14] Chen H.-F., Duncan T. E., Pasik-Duncan B. A, Kiefer-Wolfowitz algorithm with randomized differences. *IEEE Trans. Automat. Contr.*, vol. 44, No 3, 1999, pp. 442–453.

[15] Guo L., Stability of recursive stochastic tracking algorithms, *SIAM J. Control and Optimization*, vol. 32, No 5, 1994, pp. 1195–1225.

[16] Gurevich L., Vakhitov A., "SPSA Algorithm for Tracking", *In Proc. 12th International Student Olympiad on Automatic Control (Baltic Olympiad)*, 2008, pp. 52–57.

[17] Guo L., Ljung L. Performance analysis of general tracking algorithms, *IEEE Trans. Automat. Contr.*, vol. 40, No. 8, 1995, pp. 1388-1402.

[18] Spall J.C., *Introduction to Stochastic Search and Optimization*. Wiley & Sons, New Jersey; 2003.

[19] Katkovnik V. Ya., Khejsin V. E., Dynamic stochastic approximation of polynomial drifts, *Automat. Remote Control*, vol. 40, No 5, 1979, pp. 700–708.

[20] Polyak B.T., Convergence and rate of convergence in iterative stochastic processes. I. The general case, *Automat. Remote Control*, No. 12, 1976, pp. 83–94.

[21] Shiryaev A.N. *Probability.* Springer; 1995.