2010 American Control Conference
Marriott Waterfront, Baltimore, MD, USA
June 30-July 02, 2010

Adaptive Control of SISO Plant with Time-Varying Coefficients Based on Random Test Perturbation.

Alexander Vakhitov,Vsevolod Vlasov,Oleg Granichin

***Abstract*— Indirect adaptive control problem in closed loop time-varying linear system is addressed. Small random test signals are added to the plant's inputs in order to identify and track its parameters. The noise is assumed to be unknown but bounded. It may not have good statistical properties which is the case in many applications. Simulation example of non minimum-phase plant of second order is provided.**

I. INTRODUCTION

Stochastic approximation appeared in a 1951 article in the *Annals of Mathematical Statistics* by Robbins and Monro [1] and was further developed for optimization problems by Kiefer and Wolfowitz [2]. Now this topic has wide variety of applications in such areas as signal processing, telecommunications and information technology, neural networks control. In the field of adaptive control stochastic approximation algorithms can be used as well for system identification, when the parameters of a plant are unknown [3]. In [5], [6], [7], [8] it was proposed to use random perturbation added to the plant's inputs in linear time-invariant systems for identification. This paper brings together those results with recent advances on minimum tracking of time-varying functional with SPSA-type algorithms [9]. However, due to different reasons dynamical system can be time-varying. In this case the algorithm of identification needs to track the change in system's parameters to correct the control signals appropriately.

The problem of system identification of ARX model can be formulated as a problem of linear regression [3], [8]:

$$y_t = \phi_t^T \theta_t + v_t,$$

where $y_t \in \mathbb{R}$ is an observation, $\phi_t \in \mathbb{R}^p$ is a known regressor, $\theta_t \in \mathbb{R}^p$ is a vector of unknown parameters and $v_t \in \mathbb{R}$ is a noise term. In case of ARX model, vector ϕ_t consists of plant inputs and outputs. One popular way to estimate and track θ_t is to use LMS or Kalman filtering method [3]. However, in order to get proper quality of tracking with standard LMS, one needs to assure that the regressors ϕ_t and noise v_t are weakly correlated, such that, as assumed in [25]:

$$\sup_k \| \sum_{t=k+1}^{k+n} \phi_t v_t \| \leq c\sqrt{n}.$$

It is the case if either v_t is white noise (sequence of independent random variables with zero mean) or $E\phi_t = 0$, v_t, and ϕ_t are independent. In most systems controlled by some complex rules we cannot assume that $E\phi_t = 0$, because ϕ_t contains plant inputs and outputs. Those cases are too strict for most systems.Sometimes it can not be assumed that v_t is random [4].

Random perturbation added to the plant's input can overcome both of these complexities. The idea to use random test signals to identify parameters of a dynamic system firstly was proposed by Saridis and Stein in 1968 [10]. Later, it was decided to combine an identification algorithm based on random test perturbation with stabilization algorithm which cares about stability of a system [5], [7]. An algorithm based on random perturbation similar to the SPSA algorithm [12], [13] in linear case can be applied to identify the unknown parameters of a linear plant [7]. This paper extends the algorithm to cover the case of time-varying systems using recent advances on SPSA algorithms in non-stationary stochastic optimization problems [9].

The random test perturbation algorithms were successfully applied in case of linear regression [8]. They can be used for identification of dynamic system model coefficients [16], because the model can be represented as several linear regression models with unknown but bounded additive terms, which is acceptable for the random test perturbation algorithms as it was described above. After proper parameter introduction [5], [16], [7] model of any linear plant with scalar inputs and outputs can be represented as a series of linear regression models with unknown but bounded additive noise terms. This implies that random perturbation algorithm can be used to identify any linear plant.

In previous works on identification of linear plant with random test perturbation [5], [7], [10] vanishing with time test signal was used. It is obvious that the vanishing perturbation can not help in case of time-varying system which does not converge to some state (as in [21]). From the other side, if the random perturbation signal does not vanish, then the control is not asymptotically optimal. We can only guarantee some bounds on sub-optimality of the control law with direct relation to corresponding upper bound on minimum tracking error by stochastic approximation algorithm.

In the following section we will give a problem statement, in the third section the adaptive control algorithm is presented, fourth section is devoted to identification algorithm. Simulation results are described in the fifth section , while the sixth section contains conclusions, discussion of results and open problems. The appendices contain description of parameter introduction technique which is a key to understand the correspondence linear plant control and proofs of

O. Granichin, V. Vlasov and A. Vakhitov are with Department of Mathematics and Mechanics, Saint Petersburg State University,198504 Saint Petersburg, Russia oleg_granichin@mail.ru, vsevik@mail.ru, av38@yandex.ru

the theorems about the tracking algorithm.

II. PROBLEM STATEMENT

Consider a SISO plant

$$a(\nabla, \tau_t)y_t = b(\nabla, \tau_t)u_t + v_t, \quad (1)$$

where ∇ is the time shift translation operator ($\nabla y_t = y_{t-1}$), $t \in \mathbb{N}$ is discrete time moment, y_t is plant output, u_t is plant input, v_t is noise sequence. The polynomials $a(\lambda, \tau)$ and $b(\lambda, \tau)$ are defined as

$$a(\lambda, \tau_t) = 1 + \lambda a_t^{(1)} + \ldots + \lambda^n a_t^{(n)};$$

$$b(\lambda, \tau_t) = \lambda b_t^{(1)} + \lambda^2 b_t^{(2)} + \ldots + \lambda^m b_t^{(m)},$$

and the polynomials are mutually non-concellable.

We assume that the noise is bounded and unknown:

$$|v_t| < \sigma_v. \quad (2)$$

We denote the unknown plant parameters as τ_t:

$$\tau_t = (a_t^{(1)}, a_t^{(2)}, \ldots, a_t^{(n)}, b_t^{(1)}, \ldots, b_t^{(m)})^T, \quad (3)$$

$\tau_t \in T$, where $p = n+m$, $T \subseteq \mathbb{R}^p$ is a known convex closed set of possible values of the vector (3).

Definition 2.1 The drift of the sequence of parameters $\{x_t\} \subset \mathbb{R}^q$ with constants X_i, $i = 1 \ldots q$, is formulated as

$$|x_t^{(i)} - x_{t-1}^{(i)}| < X_i \text{ for } i = 1 \ldots q \text{ and } X_i > 0.$$

The feedback of the following form exists:

$$\alpha(\nabla, \tau_t)u_t + \beta(\nabla, \tau_t)y_t = 0, \quad (4)$$

which stabilizes the plant (1) and ensures an appropriate control performance such as control optimality with respect to some criterion. Let coefficients of the polynomials

$$\alpha(\lambda, \tau_t) = 1 + \lambda\alpha_1(\tau_t) + \cdots + \lambda^p\alpha_p(\tau_t),$$

$$\beta(\lambda, \tau_t) = 1 + \lambda\beta_1(\tau_t) + \cdots + \lambda^p\beta_p(\tau_t)$$

be known and continuous functions in the set T. We remind that stabilization of feedback (the controller) (4):

$$\sup_{t \in \mathbb{N}}(|y_t| + |u_t|) < \infty \quad (5)$$

is equivalent to stability of the characteristic polynomial $g(\lambda, \tau_t) = a(\lambda, \tau_t)\alpha(\lambda, \tau_t) - b(\lambda, \tau_t)\beta(\lambda, \tau_t)$ of the closed system (1),(4).

Consider a one-to-one mapping $R : \tau_t \mapsto \theta_t$

$$\theta_t = (\theta_t^{(1)}, \ldots, \theta_t^{(p)}),$$

$$\theta_t^{(i)} = \sum_{j=0}^{i} F_{j,t}^{(i)} b_t^{(i-j+1)} = F_t^{(i)}(\nabla, \tau_t) b_t^{(i+1)}(\tau_t),$$

where $F_t^{(i)}$ is an i-th degree polynomial and $F_{j,t}^{(i)}$ is its j'th coefficient, the exact formula for $F_t^{(i)}$ is given in the Appendix. To obtain τ_t from θ_t it is needed to solve a linear system of equations $A_t\theta_t = B_t$ where A_t is $p \times p$ matrix and $B \in \mathbb{R}^p$. The system is uniquely solvable if $a_t(\lambda)$ and $b_t(\lambda)$ are mutually non-concellable [7].

Next we give a definition of stabilization of estimates' sequence which will be later used in the problem statement.

Definition 2.2[26] A sequence of mean-squared estimation errors $E\|\hat{\theta}_s - \theta_s\|^2$ has *upper bound* $L > 0$ if

$$\overline{\lim_{s \to \infty}} (E\|\hat{\theta}_s - \theta_s\|^2)^{1/2} \le L.$$

Problem 1 Build a sequence of estimates $\{\hat{\theta}_s\} \subset \mathbb{R}^p$ following the unknown vectors $\{\theta_s\}$ such that an upper bound for the mean-squared errors exists.

Problem 2 Build a sequence of estimates $\{\hat{\tau}_t^{(i)}\} \subset \mathbb{R}$ following the unknown parameters $\{\tau_t^{(i)}\}$ where $i = 1, 2, \ldots, p$ such that for each i an upper bound L_i for the mean-squared errors exists.

Theorem 2.1 If for the plant (1)

1) polynomials a_t and b_t are non-concellable;
2) drift of parameters τ_t happens according to definition 2.1 with constant A_i for the coefficient $a_t^{(i)}$ and with constant B_j for the coefficient $b_t^{(j)}$;
3) $|a_t^{(i)}| \ge a_- > 0$ for $i = 1 \ldots n$;
4) $|b_t^{(j)}| \le b_+$ for $j = 1 \ldots m$;

then exists T_k: $|\theta_t^{(k)} - \theta_{t-1}^{(k)}| < T_k$ for $k = 1 \ldots p$.

Proof 1. Consider a system of equations $A_t\theta_t = B_t$. From i-th equation, $\sum_j a_t^{ij}\theta_t^{(i)} = b_t^{(i)}$ where a_t^{ij} is an element of A_t. From here, $|\theta_t^{(i)}| \le \frac{b_+}{\sum_j a_-}$.

2. Next, consider two systems for $t-1$ and t and substract the i-th equations. We get: $\sum_j a_t^{ij}(\theta_t^{(i)} - \theta_{t-1}^{(i)}) + (\sum_j a_t^{ij} - \sum_j a_{t-1}^{ij})\theta_{t-1}^{(i)}$, $|\theta_t^{(i)} - \theta_{t-1}^{(i)}| \le (B_i + \bar{A}_i\frac{b_+}{\sum_j a_-})\frac{1}{\sum_j a_-}$, where $\bar{A}_i = \sum_{k=1}^{i} A_k$. End of proof.

Theorem 2.1 states that drift of τ_t induces drift in the same sense of definition 2.1 for θ_t.

III. ALGORITHM

The identification algorithm is iterative. The first block of operations (1-4) is performed for each time moment t. The next block (5-6) is performed after the first block only when $t = sp - 1$ for some $s \in \mathbb{N}$.

1) *Generate input*
   Compute the input u_t using the regulator (4) with estimate $\hat{\tau}_t$.
2) *Generate test perturbation and feed input to plant*
   Add random test perturbation to the input:

   $$\bar{u}_t = u_t + \Delta_{s+1}, \text{ for t=ps}, \ \bar{u}_t = u_t \text{ in other case}, \quad (6)$$

   and feed $\bar{u}_t$ to the plant.
3) *Get the plant output*
   Get the plant output y_{t+1}.
4) *If STRIPE condition is true, apply the "Stripe" algorithm*
   Correct the estimate of τ_t using the "Stripe" algorithm:

   $$\hat{\tau}_{t+1} = \text{STRIPE}(\hat{\tau}_t, y_{t+1}, y_t, \ldots, u_t, u_{t-1}, \ldots).$$

Additional steps for $t = sp-1$ if STRIPE condition was false for $t \in \{p(s-1), \ldots, ps-1\}$:

5) *Calculate* $\hat{\theta}_s$
Use available output $Y_s = (y_{p(s-1)+1}, \ldots, y_{ps})^T$ to calculate $\hat{\theta}_s$ as follows:

$$\hat{\theta}_s = \text{IDENTIFY}(\hat{\theta}_{s-1}, \Delta_s, Y_s). \quad (7)$$

6) *Calculate* $\hat{\tau}_t^R$ Calculate $\hat{\tau}_t^R$ using $\hat{\theta}_s$:

$$\hat{\tau}_{t+1} = R^{-1}(\hat{\theta}_s).$$

The STRIPE condition and STRIPE algorithm are clarified in the Appendix, the IDENTIFY algorithm is the main point of the article and it is discussed in the next section. The function $R^{-1} : \theta_t \mapsto \tau_t$ used in the last step is briefly described in section 2 and fully introduced in the appendix 1.

1) On each step we use "stripe" algorithm to stabilize the system if needed. This could be understood as a high-level way to estimate parameters on the early phase of identification and tracking, while main algorithm is not yet able to give good enough estimations to keep system stable.

2) We use reparametrization technique of one-to-one relationship $R : \tau_t \mapsto \theta_t$ so that we could estimate θ instead of τ.

3) We take p as the number of steps for one algorithm cycle. On the beginning of each cycle we add a random perturbation Δ_s to the input and after p steps calculate new value of θ for this cycle. Then we calculate corresponding estimate for τ value.

IV. Algorithm for θ_s estimation

In this section we explain the IDENTIFY algorithm used in the previous section to generate estimates of θ_s.

We denote $Y_s = (y_{p(s-1)+1}, \ldots, y_{ps})^T$, $\bar{\theta}_s = (\theta^{(1)}_{p(s-1)+1}, \ldots, \theta^{(p)}_{ps})^T$, then the following equality holds:

$$Y_s = f_s(\bar{\theta}_s) + \phi_s + \Delta_s \bar{\theta}_s, \quad (8)$$

where (see Appendix 1 for definition of $F_s^{(l)}$, $G_s^{(l)}$)

$$f_s(\bar{\theta}_s) = (f_s^{(1)}(\bar{\theta}_s), f_s^{(2)}(\theta_s), \ldots, f_s^{(p)}(\bar{\theta}_s))^T,$$

$$f_s^{(l)}(\bar{\theta}_s) = F^{(l-1)}(\nabla, \tau_t) b_t(\nabla) u_{p(s-1)+l} + G^{(l-1)}(\nabla, \tau_t) y_{ps},$$

$$\phi_s = (\phi_s^{(1)}, \phi_s^{(2)}, \ldots, \phi_s^{(p)}),\ \phi_s^{(l)} = F^{(l-1)}(\nabla, \tau_s) v_{p(s-1)+l},$$

$$l = 1, \ldots, p.$$

The following algorithm can be proposed to estimate θ_s with $\hat{\theta}_s$:

$$\hat{\theta}_s = \hat{\theta}_{s-1} - \gamma_s \Delta_s (\Delta_s \hat{\theta}_{s-1} + f_s(\hat{\theta}_{s-1}) - Y_s), \quad (9)$$

where $\gamma_s > 0$ is the algorithm step size and $\gamma_s \delta_s^2 = \nu$ for some $\nu > 0$.

Theorem 4.1 For the plant (1) with feedback (4) with drift of θ_t according to definition 2.1 with constants T_i assume that

1) $E\|\Delta_s\|^2 = \delta_s^2$,
2) correlation of random perturbation and inputs for $0 < j \le l$ is bounded $E\Delta_s u_{p(s-1)-1+k} \le C^k$,
3) plant output y_t and input u_t are bounded $|y_t| < Y$, $|u_t| < U$,
4) $0 < (1 - \gamma\delta^2)^2 + \gamma^2\delta^2 U^2 < 1$.

estimation error for the i-th component of θ_s $\hat{\theta}_s^{(i)} - \theta_s^{(i)}$ provided by the algorithm (9) has an asymptotic upper bound $L^{(i)}$, the formula for which can be found in the Appendix II.

Note According to the theorem 2.1, we can assume that drift in the sense of definition 2.1 happens with θ_t while it physically happens with τ_t.

Consider a case of linear dynamic plant of second order:

$$y_t + a_t y_{t-1} + y_{t-2} = b_t^{(1)} u_{t-1} + b_t^{(2)} u_{t-2} + v_t,\ t = 1, 2, \ldots. \quad (10)$$

We apply the following parametrization of a plant:

$$\theta_t = \begin{pmatrix} b_t^{(1)} \\ b_t^{(2)} - a_t b_t^{(1)} \\ (a_t^2 - 1) b_t^{(1)} - a_t b_t^{(2)} \end{pmatrix}. \quad (11)$$

It has a following reverse mapping:

$$\begin{pmatrix} a_t \\ b_t^{(1)} \\ b_t^{(2)} \end{pmatrix} = \begin{pmatrix} -(\theta_t^{(3)} + \theta_t^{(1)})/\theta_t^{(2)} \\ \theta_t^{(1)} \\ \theta_t^{(2)} - \theta_t^{(1)}(\theta_t^{(3)} + \theta_t^{(1)})/\theta_t^{(2)} \end{pmatrix} \quad (12)$$

Let us assume that the following conditions hold for all $n \in \mathbb{N}$:

$$|b_t^{(1)} - b_{t-1}^{(1)}| \le B_1;\ |b_t^{(2)} - b_{t-1}^{(2)}| \le B_2;\ |a_t - a_{t-1}| \le A, \quad (13)$$

$$|a_t| \le \bar{A};\ |b_t^{(1)}| \le \bar{B}_1;\ \ |b_t^{(2)}| \le \bar{B}_2;\ |b_t^{(2)} - a_t b_t^{(1)}| \ge J.$$

Theorem 4.2 For the second order plant (10) assume that all the conditions of the Theorem 4.1 hold and the sequence τ_t has drift according to definition 2.1 with constants A_i and B_j for components $a_t^{(i)}$ and $b_t^{(j)}$ respectively. Then the algorithm (9) provides estimates for τ_t which stabilize in the following sense:

$$(E|a_t - \hat{a}_t|^2)^{1/2} \le \frac{1}{J}(L^0 + L^2),$$

$$(E|b_t^{(1)} - \hat{b}_t^{(1)}|^2)^{1/2} \le L^0,$$

$$(E|b_t^{(2)} - \hat{b}_t^{(2)}|^2)^{1/2} \le \frac{2\bar{B}_1}{J} L^0 + L^1 + \frac{\bar{B}_1}{J} L^2,$$

where

$$T^{(0)} = B_1;\ T^{(1)} = B_2 + 1/2(\bar{A}B_1 + A\bar{B}_1);$$

$$T^{(2)} = \bar{B}_1 A\bar{A} + 1/2(\bar{A}^2 - 1)B_1 + 1/2(\bar{A}B_2 + A\bar{B}_2).$$

Proof of Theorem 4.2 From the conditions (13) we get the dependency of error bounds $\hat{\tau}_t^{(i)} - \tau_t^{(i)}$ from $\hat{\theta}_t = \theta_t$ and from Theorem 4.1 we get the bounds $L^{(0)}$, $L^{(1)}$, $L^{(2)}$.

V. STABILIZING ALGORITHM "STRIPE"

To solve the stabilization problem (5) the algorithm described below [23] can be used, which is to be combined with the identification and tracking algorithm described in previous section. Let us transform the equation (1) into the following form:

$$y_t + \Phi^*_{t-1}\tau_t = v_t,$$

where $\Phi_{t-1} = (y_{t-1}, y_{t-2}, \cdots, y_{t-n}, u_{t-k-1}, \cdots, u_{t-m})^T$. By boundenness of noise term (2), the inequalities

$$|y_t + \Phi_{t-1}\hat{\tau}_t| \le 2\sigma_v + \epsilon|\Phi_{t-1}|,\ t \in \mathbb{N},\ |\Phi| = \sqrt{\Phi * \Phi}$$

can be solved with respect to $\hat{\tau}_t$ for any $\epsilon \ge 0$. These inequalities provide an algorithm

$$\hat{\tau}_{t+1} = Pr_T\big(\hat{\tau}_t - \frac{\eta_t \mathbf{1}(|\eta_t| - 2\sigma_v - \epsilon|\Phi_{t-1}|)}{|\Phi_{t-1}|^2}\Phi_{t-1}\big), \quad (14)$$

$$\eta_t = y_t + \Phi^*_{t-1}\tau_t = \Phi^*_{t-1}(\hat{\tau}_t - \tau_t) + v_t.$$

Given an arbitrary initial condition $\hat{\tau}_1$, the algorith converges in finite number of steps [23].

VI. SIMULATION

As an example of calculation we used the bicyclist-robot model presented in [7]. The robot controls the turn angle of bicycle handlebars as an input and the angle of deviation from the vertical line is the system output. One can consider that while robot is riding a bicyle some environmental or internal parameters could change, e.g. wind speed, surface type or friction coefficient in front fork of bicycle.

We simulate parameters identification in the linear plant mentioned above (10). We choose $y_1 = y_2 = u_1 = u_2 = 0$, where v_t is uniformly distributed between 0 and 1. We use following parameters and bounds of possible values:

$$a_t = 3.7 + 3 \cdot 10^{-4}t,\ b_t^{(1)} = 6.4 - 3 \cdot 10^{-4}t,$$

$$b_t^{(2)} = -8.0 + 3 \cdot 10^{-4}t,$$

$$\tau_t = \begin{pmatrix} a_t \\ 1 \\ b_t^{(1)} \\ b_t^{(2)} \end{pmatrix} \in T = [2, 10] \times 1 \times [1, 10] \times [-10, 0] \subset \mathbb{R}^4.$$

Under the given assumptions the mapping $\Theta(\cdot) : T \to \Theta = \theta(T) \subset \mathbb{R}^4$ is reversible (12). As the control goal we consider system stabilization and deviaton limit minimization of the system output absolute value for the "worst" possible noise sequence:

$$\sup_{\substack{v_t \le 1 \\ t=1,2,\ldots}} \overline{\lim_{t\to\infty}} |y_t| \to \min$$

When $|b_t^{(1)}/b_t^{(2)}| \le 1$ the plant is non-minimum phase and following regulator is needed to keep values of input and outputs small:

$$c_1(\tau) = \frac{b_t^{(2)} b_t^{(1)} - a_t(b_t^{(2)})^2}{(b_t^{(1)})^2 + (b_t^{(2)})^2 - a_t b_t^{(1)} b_t^{(2)}},$$

$$d_0(\tau) = \frac{b_t^{(2)} + a_t b_t^{(1)} - a_t(b_t^{(2)})^2}{(b_t^{(1)})^2 + (b_t^{(2)})^2 - a_t b_t^{(1)} b_t^{(2)}},$$

$$d_1(\tau) = \frac{b_t^{(1)} - a_t b^{(2)}}{(b_t^{(1)})^2 + (b_t^{(2)})^2 - a_t b_t^{(1)} b_t^{(2)}},$$

$$u_t + c_1(\tau)u_{t-1} = d_0(\tau)y_t + d_1(\tau)y_{t-1}.$$

This regulator is proved to be optimal for stationary case [24] and we consider its use appropriate for small enough parameters drift. Let $\{\Delta_s\}_{s\in\mathbb{N}}$ - be a set of random independent test perturbations possesing with equal probabilities values of $\pm\Delta_0, \Delta_0 = 0.3$, and the sequence of control inputs be formed by:

$$u_t = \overline{u}_t, t = 1, 2, \ldots, 99,$$

$$u_{70+30s+i} = \overline{u}_{70+30s+i}, i = 1, 2, \ldots, 29,$$

$$u_{70+30s} = \overline{u}_{70+30s} + R_s\Delta_s,$$

$$R_n = 20(1 + |y_{30s}| + |y_{30s-1}| + |\overline{u}_{30s-1}|), n = 1, 2, \ldots,$$

where test perturbation $R_s\Delta_s$ is added to control only on each 30th step starting with 100th, and the control $\overline{u}_t$ itself is determined by the regulator mentioned above and using current parameter estimation $\hat{\tau}_t$:

$$\overline{u}_t + c_1(\hat{\tau}_t)\overline{u}_{t-1} = d_0(\hat{\tau}_t)y_t + d_1(\hat{\tau}_t)y_{t-1}.$$

We start with estimation $a = 4, b^{(1)} = 5, b^{(2)} = -6$. These values are selected in order to have a visual plot showing both identifying and tracking. For "stripe" algorithm we take $\epsilon = 0.1$. Each step we use "stripe" algorithm to form new estimation of $\hat{\tau}_t$ if system became unstable. We also use identification and tracking algorithm (9) three steps after each test perturbation was sent (in case "stripe" did not change estimation value since that moment):

$$\hat{\theta}_s = \hat{\theta}_{s-1} - \gamma_s\Delta_s\Gamma(\Delta_s\hat{\theta}_{s-1} + f_s(\hat{\theta}_{s-1}) - Y_s),$$

$$\Gamma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix},$$

$$\gamma_s = \frac{\alpha_n}{\Delta^2}, \alpha_n = \max(\alpha_0, \frac{1}{n}), \alpha_0 = 0.5,$$

where Γ is a positive-definite matrix used to track different members of θ with different speed. Matrix Γ firstly introduced here does not have strong impact on the proof of theorem but allow to have much better experimental results.

Figures 1,2,3 present typical parameter estimation for $a, b^{(1)}, b^{(2)}$. Time is scaled up for first 10 steps while estimation changes dramatically. The "stripe" algorithm computes rough estimations during first steps. After 100th step when test perturbation is turned on it produces even better results. After 150th step it is not changing estimation anymore. During 1000 steps plant parameters shift by 0.3 and estimates follow these changes.

As can be seen at figure 4, while tracking estimation error does not exceed 0.15 for all paramaters. In this case the error is almost as big as parameters drift, but it should be understood that the same bounds for estimation error will be still kept even for ten times bigger steps number. We purposely use these data to make figures more visual.

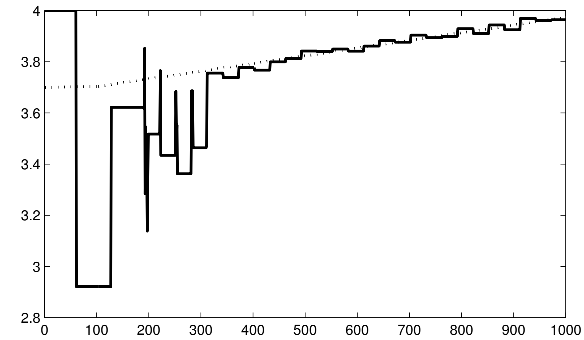

Fig. 1. Identifying and tracking a_t

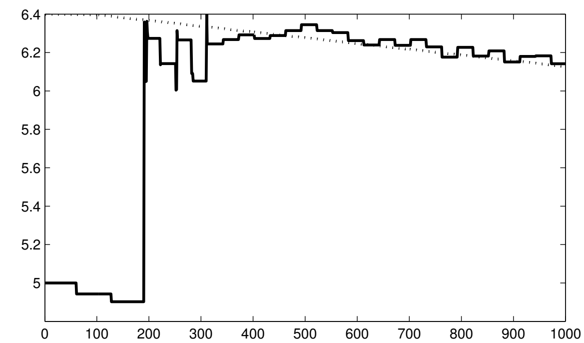

Fig. 2. Identifying and tracking $b_t^{(1)}$

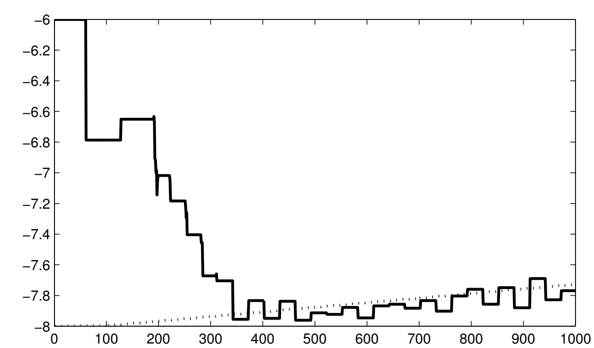

Fig. 3. Identifying and tracking $b_t^{(2)}$

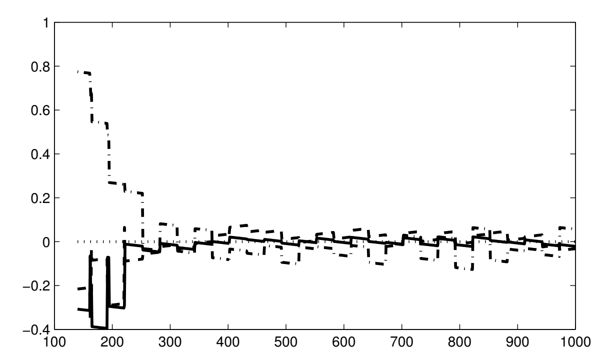

Fig. 4. Estimation errors $\hat{a}_t - a_t, \hat{b}_t^{(1)} - b_t^{(1)}, \hat{b}_t^{(2)} - b_t^{(2)}$

VII. Conclusions

The indirect adaptive control of a time-varying system is a well-known problem. However, traditional identification methods are not applicable to the general case of plant identification in presense of unknown but bounded noises and arbitrary feedback control. The use of random perturbation makes it feasible. The combination of stabilization and identification algorithm was presented in the paper.

The error bounds can be sufficiently improved with introduction of adaptive gain sequence, which could adjust to feedback and drift of plant's parameters. Another drift models could be tested as well.

VIII. Appendix 1

Next we will introduce a technique of reparametrization of (1) [7]. Consider an equation

$$F^{(l)}(\lambda, \tau_t) a(\lambda, \tau_t) + \lambda^{l+1} G^{(l)}(\lambda, \tau_t) = 1, \qquad (15)$$

where $F^{(l)}(\lambda, \tau_t)$ and $G^{(l)}(\lambda, \tau_t)$ are polynomials with respect to λ and the degree of $F^{(l)}(\lambda, \tau_t)$ is less than or equal to l. The coefficients $F^{(l)}(\lambda, \tau_t)$ can be determined by the coefficients of $a(\lambda, \tau_t)$ from equations

$$F_0^{(l)} = 1, \ \frac{\partial^i}{\partial \lambda^i} F^{(l)}(\lambda, \tau_t) a(\lambda, \tau_t) = 0.$$

Then the coefficients of $G^{(l)}(\lambda, \tau_t)$ can be found as

$$G^{(l)}(\lambda, \tau_t) = \frac{1 - F^{(l)}(\lambda, \tau_t) a(\lambda, \tau_t)}{\lambda^{l+1}}.$$

If we apply the operator $F^{(l)}(\lambda, \tau_t)$ to both sides of the plant equation (1), we get the following equation:

$$y_t = G^{(l)}(\lambda, \tau_t) y_{t-l-1} + F^{(l)}(\lambda, \tau_t) b(\lambda, \tau_t) u_t + F^{(l)}(\lambda, \tau_t) v_t. \qquad (16)$$

Let us break the set of natural numbers $\mathbb{N}$ into non-intersecting subsets N_p of $p = m + n$ numbers:

$$N_s = \{p(s-1)+1, p(s-1), \ldots, ps\}.$$

For each $t = p(s-1)+l$,$l = 1 \ldots p$ we get an equation (16). Let us denote the coefficient of $u_{p(s-1)}$ on the right-hand side of (16) as $\theta_s^{(l+1)}$. Then, if we define $\theta_s^{(i)} = 0, \ i < 0$, then the following holds:

$$a(\nabla, \tau_t) \theta_s^{(l)} = b_t^l, \ t = p(s-1) + l. \qquad (17)$$

In the matrix form the condition (17) can be rewritten as $\mathcal{A}(\tau_t)\theta_s = \mathcal{B}(\tau_t)$, we do not show the exact form of matrices due to lack of space. The one-to-one relationship between θ_t and τ_t is thoroughly described in [7].

IX. Appendix II.

Lemma 1. [9] If $v_n^2 \leq k v_{n-1}^2 + h v_{n-1} + l \ \ n \in \mathbb{N}, k \in (0,1)$. Then $v_n^2 \leq (k + \epsilon/2)^n v_0^2 + (\frac{h}{2(1-k)}(1 + \sqrt{1 + 4l(1-k)h^{-2}}))^2$, or $\overline{\lim}_{n\to\infty} v_n \leq \frac{h}{2(1-k)}(1 + \sqrt{1 + 4l(1-k)h^{-2}})$, where $\epsilon = \frac{h^2}{2l}(\sqrt{1 + 4l(1-k)h^{-2}} - 1)$.

Proof of Theorem 4.1. We will do the proof of asymptotic bounds for $|\hat{\theta}_s^{(i)} - \theta_s^{(i)}|^2$ using mathematical induction by i.

The base for the induction is $i = 1$. Let us denote as E_s the following conditional expectation:

$$E_s = E\{\cdot|\hat{\theta}_{s-1}, \theta_s\},\ \bar{E}_s = E\{\cdot|\hat{\theta}_{s-1}, \theta_{s-1}\}.$$

Let us analyze the estimation error of the initial component of θ_s. Using the fact that $F^{(0)} = 1$ we get the following:

$$|\hat{\theta}_s^{(1)} - \theta_{ps}^{(1)}|^2 = |\hat{\theta}_{s-1}^{(1)} - \theta_{ps}^{(1)} - \gamma\Delta_s(\Delta_s\hat{\theta}_{s-1}^{(1)} + f_s(\hat{\theta}_{s-1})^{(1)} - -Y_s^{(1)})|^2 = |\hat{\theta}_{s-1}^{(1)} - \theta_{ps}^{(1)} - \gamma|\Delta_s|^2(\hat{\theta}_{s-1}^{(1)} - \theta_{ps}^{(1)}) + +\gamma\Delta_s(f_s^{(1)}(\hat{\theta}_{s-1})^{(1)} - f_s^{(1)}(\theta_{p(s-1)}) - \phi_s^{(1)})|^2$$

Note that $f_s^{(1)}(\theta_{p(s-1)}) = \theta_{p(s-1)}^{(1)} u_{p(s-1)}$, so $f_s^{(1)}(\hat{\theta}_{s-1})^{(1)} - f_s^{(1)}(\theta_{p(s-1)}) = (\theta_{p(s-1)}^{(1)} - \hat{\theta}_s^{(1)})u_{p(s-1)}$ due to definition of f_s, and $\phi_s^{(1)} = v_{p(s-1)+1}$. In the same time due to independence of Δ_s and $\theta_s, \hat{\theta}_{s-1}, v_{p(s-1)}$ $E_s\Delta_s(f_s(\hat{\theta}_{s-1})^{(1)} - f_s(\theta_{p(s-1)}) - \phi_s) = 0$, so

$$E_s|\hat{\theta}_s^{(1)} - \theta_s^{(1)}|^2 = (\hat{\theta}_{s-1}^{(1)} - \theta_s^{(1)})^2(1 - \gamma\delta^2)^2 + +\gamma^2\delta^2((\hat{\theta}_{s-1}^{(1)} - \theta_s^{(1)})u_{p(s-1)} + \sum_{i=2}^{m}(\hat{b}_t^i - b_t^i)u_{p(s-1)-i} + +(\hat{a}(\nabla, \hat{\tau}_t) - a(\nabla, \tau_t))y_{p(s-1)} - v_{p(s-1)+1})^2,$$

where $\hat{b}(\nabla, \hat{\tau}_t)$ and $\hat{a}(\nabla, \hat{\tau}_t)$ are the polynomials with corresponding coefficients taken from estimate $\hat{\tau}_t$.

Notice that $(\sum_{i=2}^{m}(\hat{b}_t^i - b_t^i)u_{p(s-1)-i} + (\hat{a}(\nabla, \hat{\tau}_t) - a(\nabla, \tau_t))y_{p(s-1)} - v_{p(s-1)+1}) \leq U\sum_{i=2}^{m}\tilde{B}_i + Y\sum_{i=1}^{n}\tilde{A}_i + \sigma_v = P_1$. Then the previous inequality can be continued as

$$\ldots \leq (\hat{\theta}_{s-1}^{(1)} - \theta_{ps}^{(1)})^2((1 - \gamma\delta^2)^2 + \gamma^2\delta^2U^2) + +2\gamma^2\delta^2(\hat{\theta}_{s-1}^{(1)} - \theta_{ps}^{(1)})P_1 + \gamma^2\delta^2P_1^2.$$

Then, let us analyze the drift term, taking expectation $\bar{E}_s$ conditioned on $\hat{\theta}_{s-1}, \theta_{s-1}$:

$$\bar{E}_s|\hat{\theta}_s^{(1)} - \theta_{ps}^{(1)}|^2 \leq (\hat{\theta}_{s-1}^{(1)} - \theta_{p(s-1)}^{(1)})^2((1 - \gamma\delta^2)^2 + \gamma^2\delta^2U^2) + +2|\hat{\theta}_{s-1}^{(1)} - \theta_{p(s-1)}^{(1)}|(((1 - \gamma\delta^2)^2 + \gamma^2\delta^2U^2)T_1 + \gamma^2\delta^2UP_1) + T_1^2 + +\gamma^2\delta^2P_1^2 + 2\gamma^2\delta^2UP_1T_1.$$

Therefore, if the inequality

$$0 < (1 - \gamma\delta^2)^2 + \gamma^2\delta^2U^2 < 1$$

holds, the estimation errors for the first component of θ_s have asymptotic bound, which is according to the Lemma 1 with $k = (1 - \gamma\delta^2)^2 + \gamma^2\delta^2U^2$, $h = 2(((1 - \gamma\delta^2)^2 + \gamma^2\delta^2U^2)T_1 + \gamma^2\delta^2UP_1)$, $l = T_1^2 + \gamma^2\delta^2P_1^2 + 2\gamma^2\delta^2UP_1T_1$.

Let us prove the induction step, which is as follows: $i \rightarrow i + 1$. Let us suppose that the asymptotic bound for the j-th component is L^j, $j = 1 \ldots i$ so that:

$$\overline{\lim_{n\rightarrow\infty}}(E|\hat{\theta}_s^{(j)} - \theta_{ps}^{(j)}|^2)^{1/2} \leq L^{(j)}.$$

For the component $i + 1$ of the estimation error vector analogous bounds hold, with $k = (1 - \gamma\delta^2)^2 + \gamma^2\delta^2U^2$, $h = 2(((1 - \gamma\delta^2)^2 + \gamma^2\delta^2U^2)T_{i+1} + \gamma^2\delta^2UP_{i+1})$, $l = T_{i+1}^2 + \gamma^2\delta^2P_{i+1}^2 + 2\gamma^2\delta^2UP_{i+1}T_{i+1}$.

So, the value L^{i+1} is bounded. The induction is finished and the proof as well.

REFERENCES

[1] Robbins H., Monro S., A Stochastic Approximation Method, *Annals of Mathematical Statistics* vol 22, No. 3, September 1951, pp. 400-407.

[2] Kiefer J, Wolfowitz J, Stochastic estimation of the maximum of a regression function, *Annals of Mathematical Statistics*, vol 23, No. 3, September 1952, pp. 462–466.

[3] Ljung L., *System Identification*. Prentice Hall; 1999.

[4] Milanese M., Vicino A. Optimal estimation theory for dynamic systems with set membership uncertainty: an overview, *Automatica*, Vol. 27, No. 6, 1991. pp. 997–1009.

[5] Granitchin O., Fomin V. Adaptive control with test signals. *Automat. Remote Control,* No. 2, 1986, Pp. 100–112.

[6] Chen H.F., Guo L. Convergence rate of least squares identification and adaptive control for stochastic systems, *Int. J. of Control*, vol. 44, 1986, pp. 1459–1476.

[7] Fomin V.N., Granitchin O.N. Adaptive control with test signals, *In Proceedings of 6th Saint-Petersburg Workshop on Adaptive Systems Theory*, 1999.

[8] Granichin O. N., Linear regression and filtering under nonstandard assumptions (arbitrary noise). *IEEE Trans. Automat. Contr.*, vol. 49, No 10, 2004, pp. 1830–1837.

[9] Granichin O., Gurevich L., Vakhitov A. Discrete-Time Minimum Tracking Based on Stochastic Approximation Algorithm With Randomized Differences, *In Proceedings of 48th Conference on Decision and Control*, Shanghai, P.R. China, 2009.

[10] Saridis G.M., Stein G. A new algorithm for linear system identification, *IEEE Trans. Automat. Control*, vol 13, No. 4, pp.584–592,1968.

[11] Spall J.C., *Introduction to Stochastic Search and Optimization*. Wiley & Sons, New Jersey; 2003.

[12] Spall J. C. Multivariate stochastic approximation using a simultaneous perturbation gradient approximation. *IEEE Trans. Automat. Contr.*, vol. 37, 1992, pp. 332–341.

[13] Granichin O. N., A stochastic recursive procedure with dependent noises in the observation that uses sample perturbations in the input, *Vestnik Leningrad Univ. Math.*, vol. 22, No 1(4), 1989, pp. 27–31.

[14] Granichin O. N., Procedure of stochastic approximation with disturbances at the input, *Automat. Remote Control*, vol. 53, No 2, part 1, 1992, pp. 232–237.

[15] Polyak B. T., Tsybakov A. B., Optimal order of accuracy for search algorithms in stochastic optimization, *Problems of Information Transmission*, vol. 26, No 2, 1990, pp. 126–133.

[16] Granichin O. N., Polyak B. T., *Randomized Algorithms of Estimation and Optimization Under Almost Arbitrary Noises.* Moscow: Nauka; 2003.

[17] Spall J. C., "Developments in stochastic optimization algorithms with gradient approximations based on function measurements" *Proc. of the Winter Simulation Conf.*,1994, pp. 207–214.

[18] Popkov A. Yu., Gradient methods for nonstationary unconstrained optimization problems. *Automat. Remote Control*, No. 6, 2005, pp. 883–891.

[19] Polyak B. T., *Introduction to Optimization.* New York: Optimization Software; 1987.

[20] Kushner H., Yin G., *Stochastic Approximation and Recursive Algorithms and Applications.* Springer; 2003.

[21] J.C. Spall, J. A. Cristion. Model-Free Control of Nonlinear Stochastic Systems with Discrete-Time Measurements // IEEE Transactions on Automatic Control, vol. 43, No. 9, Sept. 1998, pp. 1198-1211.

[22] Borkar V. S. *Stochastic Approximation. A Dynamical Systems Viewpoint.* Cambridge University Press; 2008.

[23] V. Fomin, A. Fradkov, and V. Yakubovich. *Adaptive control of dynamic systems*. Nauka, Moscow, 1981.

[24] Barabanov A.Ye., Granichin O.N. Optimal controller for linear plant with bounded noise // Automation and Remote Control, 1984, v. 45, No.5, part 1, pp. 578-584.

[25] Guo L., Ljung L., Wang G.J. Necessary and sufficient conditions for stability of LMS, *IEEE Trans. Automat. Contr.*, vol. 42, No. 6, 1997, pp. 761-770.

[26] Guo L., Ljung L. Performance analysis of general tracking algorithms, *IEEE Trans. Automat. Contr.*, vol. 40, No. 8, 1995, pp. 1388-1402.