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SPSA With a Fixed Gain for Intelligent Control in Tracking Applications

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***Abstract*— Simultaneous perturbation stochastic approximation (SPSA) algorithm is also often referred as a Kiefer-Wolfowitz algorithm with randomized differences. Algorithms of this type are widely applied in field of intelligent control for optimization purposes, especially in a high-dimensional and noisy setting. In such problems it is often important to track the drifting minimum point, adapting to changing environment. In this paper application of the fixed gain SPSA to the minimum tracking problem for the non-constrained optimization is considered. The upper bound of mean square estimation error is determined in case of once differentiable functional and almost arbitrary noises. Numerical simulation of the estimates stabilization for the multidimensional optimization with non-random noise is provided.**

I. INTRODUCTION

The Kiefer-Wolfowitz algorithm with random differences belongs to the class of SPSA-type algorithms. Algorithms of this type have found numerous applications in control theory and become vital part for many intelligent control applications, such as neural networks learning, clustering, fuzzy programming, etc. In the area of fuzzy logic SPSA is used for solving following fuzzy programming models: fuzzy expected value model, fuzzy chance-constrained programming model and fuzzy dependent-chance programming model [1]. It also often uses fuzzy programming to estimate the value of the cost function (see for example [2]). In neural networks SPSA could be either applied during the learning process [3]–[5], or it can take neural network as a function approximation and optimize it (see for example [6]). In case of intelligent control time-varying systems are often considered, and because of that we would like to present here the modification of the SPSA-type algorithm which suits for such problems. For the Kiefer-Wolfowitz algorithm with randomized differences with a fixed step size we prove stabilization of its estimates while tracking the drifting minimum point. However, this result is very general and can be applied in other areas as well.

Problem of functional optimization arises in many practical cases. While in some cases extreme points could be found analytically, many engineering applications deal with unknown functional, which can only be measured in selected points with possible noise. In some cases functional itself could vary over time and its extreme points could drift. In this case problem setting could be different, depending on goals of optimization and possible measurements. In general, there are two different variants of a function behavior over time - it has a limit function, to which it tends when time goes to infinity, or there is no such function [7]. In this paper we consider the second variant.

Non-stationary optimization problems can be described in discrete or continuous time. In our paper we consider only discrete time model. Let $f(x, n)$ be a functional we are optimizing at the moment of time n ($n \in \mathbf{N}$). In book [8] Newton method and gradient method are applied to problems like that, but they are applicable only in case of two times differentiable functional and $l < \nabla^2 f_k(x) < L$. Both methods require possibility of direct measurement of gradient in arbitrary point.

In real world measurement always contains noise. Sometimes the algorithms that perfectly solve the problem on paper do not provide good estimates in practical cases. Robustness is important in engineering applications. For problems with noise the Robbins-Monro and Kiefer-Wolfowitz stochastic approximation algorithms were developed in 1950s. The history of development of such algorithms is described in [9], [10]. Common approach used in these algorithms can be formalized in a following way:

$$\hat{\theta}_{n+1} = \hat{\theta}_n - \alpha_n \hat{g}_n(\hat{\theta}_n), \quad (1)$$

where $\{\hat{\theta}_n\}$ —is the sequence of extreme points estimates generated by algorithm, g_n — pseudo-gradient (replacing the gradient from Newton method). Pseudo-gradient has to approximate the true gradient. The important properties of algorithms described in this form are simplicity and recurrence. Because of these properties they are often applied in different areas.

Algorithms of the SPSA-type with one or two measurements per each iteration appeared in papers of different researchers in the end of the 1980s [11]–[14]. Later in the text we will refer to this class of algorithms as SPSA for simplicity. These algorithms are known for their applicability to problems with almost arbitrary noise [10]. The measurement noise should be bounded and only slightly correlated with perturbation on each iteration. Moreover, the number of measurements made on each iteration is only one or two and is independent from the number of dimensions of the state space d. This property sufficiently increases the rate of convergence of the algorithm in multidimensional case ($d >> 1$), comparing to algorithms, that use direct estimation of gradient, that requires $2d$ measurements of function in case if direct measurement of function gradient is impossible. Detailed review of development of such methods is provided in [10], [15].

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Stochastic approximation algorithms were initially proven in case of the stationary functional. The gradient algorithm for the case of minimum tracking is provided in [8], however the stochastic setting is not discussed there. Further development of these ideas could be found in paper [7], where conditions of drift pace were relaxed. The book [9] uses the ordinary differential equations (ODE) approach to describe stochastic approximation. It addresses the issue of applications of stochastic approximation to tracking and time-varying systems in a following way: it is proven there that when the step size goes to zero in the same time as the number of the algorithm's iterates over a finite time interval tends to infinity, then the minimum estimates tend to true minimum values. This is not the case here, since we consider the number of iterates per unit of time to be fixed.

In the most of the cases, SPSA algorithm applied with coefficients α_n and β_n tending to zero. This is not applicable however in tracking case, since the dynamic nature of the observed system. In our work we use fixed gain version of algorithm. Coefficients α and β are chosen a-priory and not changed during the execution of algorithm. Properties of fixed gain algorithms for tracking under very general conditions are discussed in [16]. The fixed gain SPSA applied to discrete set optimization problem is considered in [17]. However, in both of these articles all proofs are given only for limited class of noise. In our paper we proof the applicability of algorithm in conditions of almost arbitrary noise.

In this paper we consider application of simultaneous perturbation stochastic approximation algorithm to the problem of tracking of the functional minimum. SPSA algorithm does not rely on direct gradient measurement and is more robust to non-random noise than gradient-based methods mentioned earlier. The most closely case was studied in [18], but we do not use the ODE approach and we establish more wide conditions for the estimates stabilization. In the following section we will give the problem statement that is more general than in [7], [8], in the third section we provide the algorithm and prove its estimates mean squared stabilization. In the last section we illustrate the algorithm, applying it to minimum tracking in a particular system.

II. Problem Statement

Consider the problem of minimum tracking for average risk functional:

$$f(x,n) = E_w\{F(x,w,n)\} \to \min_x, \qquad (2)$$

where $x \in \mathbf{R}^d$, $w \in \mathbf{R}^p$, $n \in \mathbf{N}$, $E_w\{\cdot\}$ — mean value conditioned on the minimal σ-algebra in which w is measurable.

The goal is to estimate θ_n — minimum point of functional $f(x,n)$, changing over time:

$$\theta_n = argmin_x f(x,n).$$

Let us assume that on the iteration we can do a following measurement:

$$y_n = F(x_n, w_n, n) + v_n,\ n = 1, 2, \ldots \qquad (3)$$

where x_n are arbitrary measurement points chosen by algorithm, w_n are random values, that are non-controlled uncertainty and v_n are observation noises.

Time in our model is discrete and implemented in number of iteration n.

To define the quality of estimates we will use the following definition:

Definition *1:* [19] A sequence of vector estimates $\{\hat{\theta}_k, k \geq 0\}$ of true vector sequence $\{\theta_k, k \geq 0\}$ defined on the basic probability space $\{\Omega, \mathcal{F}, P\}$ is called L_P-stable ($p > 0$) if

$$\sup_{k\geq 0} E^{1/p} \|\theta_k - \hat{\theta}_k\|^p = \bar{L} < \infty.$$

Further we will consider generation of sequence of estimate $\{\hat{\theta}_n\}$ for problem (2), for which there exists some $\bar{L} < \infty$ satisfying the definition 1 for $p = 2$, in following conditions.

We will assume that drift of the minimum point is limited in following sense:

$$(\mathbf{A}) \quad \|\theta_n - \theta_{n-1}\| \leq A.$$

Function $f(\cdot, n)$ is a strictly convex function for each n:

$$(\mathbf{B}) \quad \langle \nabla f(x,n), x - \theta_n \rangle \geq \mu \|x - \theta_n\|^2.$$

Gradient $\nabla F(\cdot, w, n)$ is Lipschitz with constant M, $\forall n, \forall w$:

$$(\mathbf{C}) \quad \|\nabla F(x,w,n) - \nabla F(y,w,n)\| \leq M\|x - y\|; \quad E_n \|\nabla_{\theta_n} F(\theta_n, w, n)\|^2 \leq B < \infty$$

(D) Average difference of function $F(x, \cdot, n)$ in any point x for moments n and $n+1$ is limited in a following way:

$$E_{w_1,w_2} |F(x, w_1, n+1) - F(x, w_2, n)|^2 \leq C\|x - \theta_n\|^2 + D.$$

(**E**) Local Lebesgue property for the function $\nabla F(w,x)$: $\forall x \in \mathbb{R}^d\ \exists$ neigbourhood U_x such that $\forall x' \in U_x$ $\|\nabla F(w,x)\| < \Phi_x(w)$ where $\Phi_x(w) : \mathbb{R}^p \to \mathbb{R}$ is integrable by w: $E_w\{\Phi_x(w)\} < \infty$

The last condition is necessary for the commutation of differentiation and integration operations, that is used to change order of expectation and gradient in the proof of the theorem. For more discussions about such properties see [20].

(**F**) For the observation noise v_n the following conditions are satisfied:

$$|v_{2n} - v_{2n-1}| \leq \sigma_v,$$

or if it has statistical nature then:

$$E\{|v_{2n} - v_{2n-1}|^2\} \leq \sigma_v^2.$$

Here we should make several notes: 1). Sequence $\{v_n\}$ could be of non-statistical but unknown deterministic nature. 2). Constraint (**A**) allows both random and deterministic drift. In certain cases Brownian motion could be described without tracking. Tracking is needed when there is both determined and non-determined aspects of drift. Similar condition is introduced in [8], it is slightly relaxed in [7]. For example it could be relaxed in a following way:

$$(\mathbf{A}') \quad \theta_n = A_1\theta_{n-1} + A_2 + \xi_n,$$

where ξ_n is random value.

In this paper we will only consider drift constraints in form **(A)**. Mean square stabilization of estimation under condition **(A)** implies its applicability to wide variety of problems.

III. ALGORITHM

In this section we are introducing a modification of SPSA algorithm provided by Chen et al [21], which takes one perturbed and one non-perturbed measurement on each step.

Let perturbation sequence $\{\Delta_n\}$ be an independent sequence of Bernoulli random vectors, with component values $\pm 1/\sqrt{d}$ with probability $\frac{1}{2}$. Let vector $\hat{\theta}_0 \in \mathbb{R}^d$ be the initial estimation. We will estimate a sequence of minimum points $\{\theta_n\}$ with sequence $\{\hat{\theta}_n\}$ which is generated by the algorithm with fixed stepsize:

$$\begin{cases} x_{2n} = \hat{\theta}_{2n-2} + \beta\Delta_n,\ x_{2n-1} = \hat{\theta}_{2n-2}, \\ y_n = F(x_n, w_n, n) + v_n, \\ \hat{\theta}_{2n} = \hat{\theta}_{2n-2} - \frac{\alpha}{\beta}\Delta_n(y_{2n} - y_{2n-1}), \\ \hat{\theta}_{2n-1} = \hat{\theta}_{2n-2}. \end{cases} \tag{4}$$

(**G**) We will further assume that random values Δ_n generated by algorithm are not dependent on $\hat{\theta}_k$, w_k , $\hat{\theta}_0$ and on v_k (if they are assumed to have random nature) for $k = 1, 2, \ldots, 2n$.

IV. MEANSQUARE STABILIZATION FOR ALGORITHM ESTIMATES

Let us define $H = 2\alpha\beta M + 4A + 12\frac{\alpha^2}{\beta^2}AC$ Denote $L = 8A^2 + 12\alpha^2(M^2\beta^2 + B) + 6\frac{\alpha^2}{\beta^2}(CA^2 + D + \sigma_v^2)$.

Theorem 1: Assume that conditions **(A)–(G)** on functions f, F and ∇F and values θ_n, $\hat{\theta}_n$, v_n, w_n, y_n and Δ_n are satisfied. Let K and $\delta > 0$ be constants satisfying following condition:

$$K = 2\alpha\mu - 6\frac{\alpha^2}{\beta^2}C > 0. \tag{5}$$

Let $\bar{K} = 1 - K + \delta H < 1$. Then estimates provided by the algorithm (4) stabilize in mean squares and following inequality holds:

$$E\{\|\theta_n - \hat{\theta}_n\|^2\} \leq \bar{K}^n\|\theta_0 - \hat{\theta}_0\|^2 + \frac{(L + H/\delta)(1 - \bar{K}^n)}{1 - \bar{K}}. \tag{6}$$

Note that Theorem 1 provides asymptotically effective value for the estimates: $\bar{L} = \sqrt{(L + H/\delta)/(K - \delta H)}$. This value can be minimized by δ, so taking $\delta_* = \min_\delta \bar{L}$ we achieve $\bar{L}^* = \frac{\sqrt{H^2+KL}+H}{K}$.

Conditions **(A)–(C)**,**(E)–(G)** are standard for SPSA algorithms ge. Earlier the proof of the similar theorem was given in [22] with more strict conditions. See the proof of Theorem 1 in appendix.

The condition (5) on α can be satisfied only when inequality $0 < \alpha < \frac{\mu\beta^2}{3C}$ is true. It follows from the result of the Theorem 1 that $E\{\|\theta_n - \hat{\theta}_n\|^2\} \leq O(\frac{A^2}{\alpha})$ $(\alpha \to 0)$ which leads to a simple decision rule: to presume the upper bound α should tend to zero with the same pace as A^2. α can be arbitrarily close to 0, which diminishes the effects of the gradient approximation bias and the noise.

To build the upper bound of the algorithm's estimates error using Theorem 1, it is needed to find α and β satisfying the condition (5), then to find δ which gives minimum value of the fraction $\frac{L}{1-K}$. Using the resulting bound obtained, it is possible to reduce it by applying the ideas concerning the relation of α and dreif parameters such as A and the function parameters such as M or μ. The resulting estimates behavior will be the trade-off between the tracking ability and noise sensitivity. The similar problem of algorithm parameters choosing was studied in [23] for the same algorithm but in the linear case.

V. EXAMPLES

Simple practical application of the algorithm (4) is estimation of the multidimensional moving point coordinates when only information about distance from arbitrary point to the moving point is available with additive noise. As a result of Theorem 1, the algorithm (4) provides the point estimate in case of limited drift of the point and somehow limited observation noise. Numerical examples given in this section illustrate a solution of this problem.

Consider 2 dimensional case, when drift model is described with formula $\theta_n = \theta_{n-1} + \zeta$, where ζ is uniformly distributed on the sphere: $\|\zeta\| = 1$. We will optimize function that measures square of the distance to the point $F(x, w, n) = f(x, n) = \|x - \theta_n\|^2$. Obviously, this function satisfies the conditions of the theorem. Measurements on each step are made with extra noise $y_n = f(x_n, n) + v_n$, where $v_n \in (-1, 1)$. Noise v_n is generated according the following law $v_{2i} = 1 - (i \bmod 3)$ for even steps and $v_{2i-1} = 1 - 3 * (i \bmod 7)$ for odd. In this case function parameters are $A = 0,1$, $M = 2$, $C = 0,04$, $D = 0,0004$, $B = 0$, $\mu = 2$. Then $K = 2\alpha\mu - 6\frac{\alpha^2}{\beta^2}C = 4\alpha - 0,24\frac{\alpha^2}{\beta^2} > 0$ which is true for $\alpha = 1/36$, $\beta = 1/2$. As for the estimates error bound, $H = 2\alpha\beta M + 4A + 12\frac{\alpha^2}{\beta^2}AC = 0,09$, $K \approx 0,11$, $L = 8A^2 + 12\alpha^2(M^2\beta^2 + B) + 6\frac{\alpha^2}{\beta^2}(CA^2 + D + \sigma_v^2) \approx 0,1$, $\bar{L} = \frac{\sqrt{H^2+KL}+H}{K} \approx (\sqrt{0,0081 + 0,01} + 0,09)/0,11 = 2$.

The drift of the point is shown on Fig. 1 with a bold line. Dash line illustrates estimate drift generated by (4). Initial estimate was taken as $\hat{\theta}_0 = (-5, -5)^T$. Estimation error is shown at the Fig. 2.

The second example is a hundred dimensional drift (Fig. 3). In this case standard algorithms that are based on gradient vector estimation are not applicable since it takes 200 iterations to build the approximation. So the gradient method error would be 200 multiplied by the drift speed. In our example we have drift speed 2 on each iteration and error of the method holds on level 150, which is much lower than the optimistic estimate for classical algorithm.

VI. CONCLUSION

In our work we apply the SPSA-type algorithm to the problem of extreme point tracking with almost arbitrary noise. Method does not require possibility of direct gradient measurement, needs only 2 function measurement on each iteration and once differentiable function. Drift is only assumed to be limited, which includes random and directed drift. It was proven that the estimation error of this algorithm is limited with constant value. The modeling was performed on a multidimensional case.

The authors want next to prove more precise boundaries of the estimation error. It could be also interesting to modify the algorithm to work with unknown polynomial drift, using the technique of polynomial fitting demonstrated in [24]. It would sufficiently relax the conditions of algorithm applicability. It could be also interesting to consider converging sequence of functions and apply algorithm for precise estimation of the limit minimum point

APPENDIX

Proof of the Theorem 1. *Proof:* Let $err_{n-1} = \hat{\theta}_{2n-2} - \theta_{2n-2}$, $drift_n = \theta_{2n} - \theta_{2n-2}$, $step_n = \frac{\alpha}{\beta}(y_{2n} - y_{2n-1})\Delta_n$.

According to (4) and condition **(A)** we can bound $\|err_n\| = \|\hat{\theta}_{2n} - \theta_{2n}\|$ as following:

$$\|err_n\|^2 \le \|err_{n-1}\|^2 + \|drift_n\|^2 + \|step_n\|^2 +$$

$$+2\langle drift_n, step_n\rangle - 2\langle drift_n, err_{n-1}\rangle - 2\langle step_n, err_{n-1}\rangle \le$$

$$\le \|err_{n-1}\|^2 + A^2 + \|step_n\|^2 + 2\langle step_n, drift_n\rangle - \qquad (7)$$
$$-2 < drift_n, err_{n-1}\rangle - 2\langle err_{n-1}, step_n\rangle.$$

1. According to observation model (3) for last term we have

$$-\langle err_{n-1}, step_n\rangle =$$
$$= -\langle err_{n-1}, \frac{\alpha}{\beta}\Delta_n(F(\hat{\theta}_{2n-2} + \beta\Delta_n, w_{2n}, 2n) -$$
$$-F(\hat{\theta}_{2n-2}, w_{2n-1}, 2n-1) + v_{2n} - v_{2n-1})\rangle.$$

Let us denote expectation conditioned on random values $\theta_1 \dots, \theta_{n-1},\ \hat{\theta}_1, \dots, \hat{\theta}_{n-1}$ as $E_n\{\cdot\}$. Applying this operator to last statement, we get (adding and extracting $F(\hat{\theta}_{2n-2}, w_{2n}, 2n)$) the following inequality:

$$E_n\{-\langle err_{n-1}, step_n\rangle\} = -\langle err_{n-1}, \tfrac{\alpha}{\beta}E_n\{\Delta_n$$
$$(F(\hat{\theta}_{2n-2} + \beta\Delta_n, w_{2n}, 2n) - F(\hat{\theta}_{2n-2}, w_{2n}, 2n))\}\rangle -$$
$$-\langle err_{n-1}, \tfrac{\alpha}{\beta}E_n\{\Delta_n(F(\hat{\theta}_{2n-2}, w_{2n}, 2n) -$$
$$-F(\hat{\theta}_{2n-2}, w_{2n-1}, 2n-1))\}\rangle. \qquad (8)$$

Here we use $E_n\{\Delta_n(v_{2n} - v_{2n-1})\} = 0$. Consider the difference under $E_n\{\cdot\}$ in first term (8). Taking into account presentation of $F(\hat{\theta}_{2n-2} + \beta\Delta_n, w_{2n}, 2n)$ as Taylor series, we conclude

$$\Delta_n(F(\hat{\theta}_{2n-2} + \beta\Delta_n, w_{2n}, 2n) - F(\hat{\theta}_{2n-2}, w_{2n}, 2n)) =$$
$$\Delta_n\langle\nabla F(\hat{\theta}_{2n-2} + \gamma_1\beta\Delta_n, w_{2n}, 2n), \beta\Delta_n\rangle =$$
$$\Delta_n(\langle\nabla F(\hat{\theta}_{2n-2}, w_{2n}, 2n), \beta\Delta_n\rangle - \langle\nabla F(\hat{\theta}_{2n-2}, w_{2n}, 2n) -$$
$$\nabla F(\hat{\theta}_{2n-2} + \gamma_1\beta\Delta_n, w_{2n}, 2n), \beta\Delta_n\rangle),$$

where $\gamma_1 \in (0,1)$. As a result for first term in (8), applying **(B)**, **(C)**, **(E)**, we get

$$\begin{aligned} -\langle err_{n-1}, \tfrac{\alpha}{\beta} E_n\{\Delta_n(F(\hat\theta_{2n-2}+\beta\Delta_n, w_{2n}, 2n) - \\ -F(\hat\theta_{2n-2}, w_{2n}, 2n))\}\rangle \le \\ \le -\alpha\langle err_{n-1}, \nabla f(\hat\theta_{2n-2}, 2n)\rangle + M\alpha\beta\|err_{n-1}\| \le \\ \le -\alpha\mu\|err_{n-1}\|^2 + \alpha\beta M\|err_{n-1}\|. \end{aligned} \quad (9)$$

For second term in (8), taking into account independence of Δ_n from w_{2n} and w_{2n-1}, we derive:

$$\begin{aligned} -\langle err_{n-1}, \tfrac{\alpha}{\beta} E_n\{\Delta_n(F(\hat\theta_{2n-2}, w_{2n}, 2n) - \\ -F(\hat\theta_{2n-2}, w_{2n-1}, 2n-1))\}\rangle = \\ = \langle err_{n-1}, \tfrac{\alpha}{\beta} E_n\{\Delta_n(F(\hat\theta_{2n-2}, w_{2n}, 2n) - \\ -F(\hat\theta_{2n-2}, w_{2n-1}, 2n-1))\} = 0. \end{aligned} \quad (10)$$

Finally, $-E_n\{\langle err_{n-1}, step_n\rangle\} \le -\alpha\mu\|err_{n-1}\|^2 + \alpha\beta M\|err_{n-1}\|$.
2. Consider $E_n\{\langle step_n, drift_n\rangle\}$. We will use a simple estimate as follows and reduce the step of proof to the next one:

$$2E_n\{\langle step_n, drift_n\rangle\} \le E_n\|step_n\|^2 + 4A^2.$$

3. Consider $E_n\|step_n\|^2$. Let us estimate some terms as follows:

$$\begin{aligned} E_n\|F(\hat\theta_{2n-2}+\beta\Delta_n, w_{2n}, 2n) - F(\hat\theta_{2n-2}, w_{2n}, 2n)\|^2 \\ \le \|\langle\nabla_x F(\hat\theta_{2n-2}+\gamma\beta\Delta_n, w_{2n}, 2n), \beta\Delta_n\rangle\|^2 \le \\ \le 2\big(\|\nabla_x F(\hat\theta_{2n-2}+\gamma\beta\Delta_n, w_{2n}, 2n) - \nabla_x F(\hat\theta_{2n}, w_{2n}, 2n)\|^2 + \\ \|\nabla_x F(\hat\theta_{2n}, w_{2n}, 2n)\|^2\big)\|\beta\Delta_n\|^2 \le 2(M^2\beta^2+B)\beta^2; \\ E_n\|F(\hat\theta_{2n-2}, w_{2n}, 2n) - F(\hat\theta_{2n-2}, w_{2n-1}, 2n-1)\|^2 \le \\ C\|\hat\theta_{2n-2} - \theta_{2n-1}\|^2 + D \le \\ C\|err_{n-1}\|^2 + 2AC\|err_{n-1}\| + CA^2 + D. \end{aligned}$$

Using properties **(C)**, **(D)**, we can estimate $E_n\|step_n\|^2$ as follows:

$$\begin{aligned} E_n\|step_n\|^2 \le \tfrac{\alpha^2}{\beta^2}(F(\hat\theta_{2n-2}+\beta\Delta_n, w_{2n}, 2n) \\ -F(\hat\theta_{2n-2}, w_{2n}, 2n) + F(\hat\theta_{2n-2}, w_{2n}, 2n) \\ -F(\hat\theta_{2n-2}, w_{2n-1}, 2n-1) + v_{2n} - v_{2n-1})^2 \le \\ \le 3\tfrac{\alpha^2}{\beta^2}(2\beta^2(M^2\beta^2+B) + C\|err_{n-1}\|^2 \\ +2AC\|err_{n-1}\| + CA^2 + D + \sigma_v^2) \end{aligned}$$

Summing previous inequality, evaluating term $\|2\langle drift_n, err_{n-1}\rangle\| \le 4A\|err_{n-1}\|$ and taking into account formula for H, we derive

$$\begin{aligned} E_n\{\|err_n\|^2\} \le \|err_{n-1}\|^2(1 - 2\alpha\mu + \tfrac{\alpha^2}{\beta}M + \tfrac{\alpha^2}{\beta^2}C) + \\ +2\|err_{n-1}\|H + A^2 + 2\alpha\beta AM + 2\tfrac{\alpha}{\beta}\sigma_v A + \\ +\tfrac{\alpha^2}{\beta^2}((\beta^2 M + D)^2 + \sigma_v^2 + 2(\beta^2 M + D)\sigma_v). \end{aligned}$$

Using the inequality $2H\|err_{n-1}\| \le \|err_{n-1}\|^2\delta + \frac{H^2}{\delta}$, $\forall\delta > 0$, we obtain

$$\begin{aligned} E_n\{\|err_n\|^2\} \le \|err_{n-1}\|^2(1 - 2\alpha\mu + \tfrac{\alpha^2}{\beta}M + \tfrac{\alpha^2}{\beta^2}C + \delta) + \\ +\tfrac{H^2}{\delta} + A^2 + 2\alpha\beta AM + 2\tfrac{\alpha}{\beta}\sigma_v A + \\ +\tfrac{\alpha^2}{\beta^2}((\beta^2 M + D)^2 + \sigma_v^2 + 2(\beta^2 M + D)\sigma_v). \end{aligned}$$

Futher we choose $0 < \delta < 2\alpha\mu - \frac{\alpha^2}{\beta}M - \frac{\alpha^2}{\beta^2}C$. Having such constants δ, H, L we get $E_n\{\|err_n\|^2\} \le K\|err_{n-1}\|^2 + L$. And with applying the unconditional expectation we finally have (6). ■

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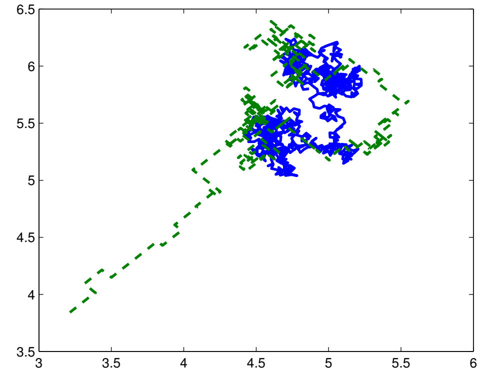

Fig. 1. 2-dimensional point drift (solid line) and estimation process (dashed line)

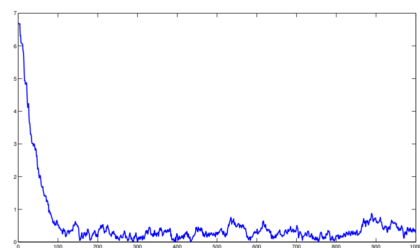

Fig. 2. 2-dimensional point drift: estimates error (solid line)

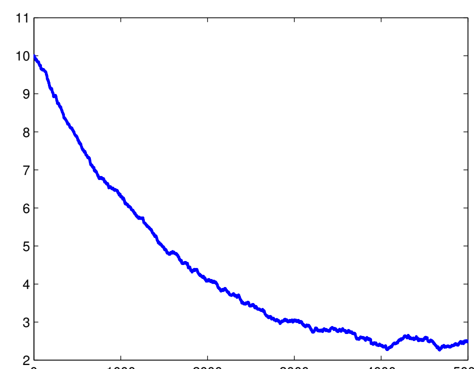

Fig. 3. 100-dimensional point drift: estimates error (solid line)