SPSA-based adaptive control: accuracy of estimates

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Abstract: Accuracy for main class of Simultaneous Perturbation Stochastic Approximation (SPSA) procedures is being researched. The model of observation is considered to be one of the most general among SPSA research. The power of moments of expectation for which the estimates of the procedure do converge is lowerized from 2 to 1 (not including lower bound). The conditions for the convergence are presented, with additional generalisations made about heavy-tailed noise and trial perturbation properties.

Key–Words: Stochastic systems, Nonlinear control, Learning control, Intelligent control

1 Introduction

The interest to complex systems with different kinds of uncertainties leads to non-classical methods of system identification and control. So called *stochastic* control is highly discussed in contemporary science. Different approaches to model systems with uncertainties do exist, each usually leading to separate class of methods of control. In this paper simultaneous perturbation stochastic approximation (SPSA) group of methods is discussed, and the set of tasks for which these methods are applicable is made wider.

Firstly we describe some typical closed-loop discrete-time system. In these terms the task typically solved by SPSA can be easily formulated. So, the system consists of object and controller. Uncertainty is being expressed by w_n and v_n sequences, first is about internal system undeterminism of behaviour, second is external noise which is added to the true object output during measurement or transfer of this discrete signal through some noisy channel.

Following [6], we can divide optimization problem statement to offline, stochastic and online classes. Offline statement is classical. The approach considered here is stochastic, where there is one function $F(x, w)$ being optimized, but it is measurable with noise. Online statement of optimization problem is new and interesting. It assumes that on each iteration new function is measured, but the cost function is formulated depending on all the functions, for example we need to find point closest in average to minimum of all the functions [6].

We continue investigations started in [3, 4] describing the types of convergence for the SPSA procedures. The convergence of $E\{\|\theta - \hat{\theta}_n\|^\rho\}$ for $\rho \in (1, 2]$ (moments of estimates of degree ρ) is being researched. Conditions from [7] are taken as sufficient for SPSA in general. Additional assumptions of existence of ρ-moment of $F(x, w)$, and some complex condition((E) in Section 2) on v_n and Δ_n should be satisfied for the results of the paper.

We try to develop SPSA algorithms theoretical framework in the very general form. Instead of assumption of triple-differentiable F as it is in [8], we consider only one-time differentiable F. Also, the simultaneous perturbation vector should be of the form $K_n(\Delta_n)$ where Δ_n is Bernoulli random vector and K_n is vector-function (kernel) with some condition on it instead os straight usage of Bernoulli random variables in [8]. The model with two kinds of uncertainty is also more general than preseted in [8] and [9], and we believe that SPSA technique becomes more useful with these generalisations.

In [3] this convergence was proved for SPSA procedure with one measurement. Using the same approach, in [4] convergence for two-measurement per iteration is proved. Here we present these results together with another procedure of this class convergence result, deliver common framework for such methods and add some generalisations, discussed in [7] but not provided in [3, 4]

2 Problem statement and SPSA algoritms

Let $F(x,w)$: $\mathbb{R}^q \times \mathbb{R}^p \to \mathbb{R}^1$ — be differentiable by the first argument function, $x_1, x_2, \ldots$ — is chosen by author of experiment sequence of points for measurement (plan of experiment), in which at every moment $n = 1, 2, \ldots$ value of a function $F(\cdot, w_n)$ is available with additive disturbances v_n.

$$y_n = F(x_n, w_n) + v_n, \tag{1}$$

where $\{w_n\}$ — uncontrolable sequrnce of random values from $\mathbb{R}^p$, having equal, but unknown distribution $\mathrm{P}_w(\cdot)$.

Problem. It is needed to construct using observations $y_1, y_2, \ldots$ a sequence of estimates $\{\hat{\theta}_n\}$ of unknown vector θ, minimizing a function

$$f(x) = \mathrm{E}_w\{F(x,w)\} = \int_{\mathbb{R}^p} F(x,w)\mathrm{P}_w(dw)$$

of average cost functional type.

Usually the problem of minimization of function $f(\cdot)$ with simpler model of observations is discussed:

$$y_n = f(x_n) + v_n, \tag{2}$$

which easily suits to the proposed scheme. More complicated model

$$y_n = w_n f(x_n) + v_n, \tag{3}$$

which suits the general model with $F(x,w) = wf(x)$, was earlier investigated in [10].

When distribution $\mathrm{P}_w(\cdot)$ is unknown, the problem discussed is outside the scope of classical optimization theory.

If measurements of function $F(x_n, w_n)$ are done in fact with some additive random centered independent noise $v_n \in \mathbb{R}$, then this extra complexity us not principal. Adding to vector w additional component v and denoting

$$\bar{w} = \begin{pmatrix} w \\ v \end{pmatrix},$$

it is possible to use instead of $F(x,w)$ another function

$$\bar{F}(x, \bar{w}) = F(x,w) + v$$

with observation scheme without additional disturbances and new common unknown distribution $\mathrm{P}_{w,v}(\cdot)$ instead $\mathrm{P}_w(\cdot)$, which was unknown before. If noise added by measurement doesn't have good statistical properties, then it is impossible to simplify the problem. It is needed to use a model with additional disturbances v_n.

Let us denote simultaneous perturbation as $\Delta_n \in \mathbb{R}^q$;

$\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of positive numbers, tending to zero; $\hat{\theta}_0 \in \mathbb{R}^q$ is a fixed initial vector. To construct the sequnces of points for measurements $\{x_n\}$ and estimates $\{\hat{\theta}_n\}$ three algorithms are proposed. First uses one observation to build an estimate:

$$\begin{cases} x_n = \hat{\theta}_{n-1} + \beta_n \Delta_n,\ y_n = F(x_n, w_n) + v_n\ , \\ \hat{\theta}_n = \hat{\theta}_{n-1} - \frac{\alpha_n}{\beta_n}\mathcal{K}_n(\Delta_n) y_n, \end{cases} \tag{4}$$

second and third use 2 observations on each iteration:

$$\begin{cases} x_{2n} = \hat{\theta}_{n-1} + \beta_n \Delta_n,\ x_{2n-1} = \hat{\theta}_{n-1} - \beta_n \Delta_n, \\ \hat{\theta}_n = \hat{\theta}_{n-1} - \frac{\alpha_n}{2\beta_n}\mathcal{K}_n(\Delta_n)(y_{2n} - y_{2n-1}), \end{cases} \tag{5}$$

$$\begin{cases} x_{2n} = \hat{\theta}_{n-1} + \beta_n \Delta_n, \qquad x_{2n-1} = \hat{\theta}_{n-1}, \\ \hat{\theta}_n = \hat{\theta}_{n-1} - \frac{\alpha_n}{\beta_n}\mathcal{K}_n(\Delta_n)(y_{2n} - y_{2n-1}). \end{cases} \tag{6}$$

In all three algorithms some vector-functions (kernels) are used: $\mathcal{K}_n(\cdot) : \mathbb{R}^q \to \mathbb{R}^q$,

which satisfy together with distributions of simultaneous perturbation $\mathrm{P}_n(\cdot)$ the conditions:

$$\int \mathcal{K}_n(x)\mathrm{P}_n(dx) = 0, \quad \int \mathcal{K}_n(x) x^{\mathrm{T}} \mathrm{P}_n(dx) = \mathrm{I}, \tag{7}$$

where I is a q-dimensional unit matrix.

Algorithm (4) with function $\mathcal{K}_n(\Delta_n) = \Delta_n$ was primary founded by O. N. Granichin in the paper [11] for constructing a sequence of estimates, well-founded in almost arbitrary noise in observations. B. T. Polyak and A. B. Tsybakov investigated in [12] both algorithms (4) and (5) with vector-function $\mathcal{K}_n(\cdot)$ of general form in situation of uniform testing perturbation and with assumption about independency and centralisation of observation noise. J. Spall [13] used algorithm (5) in case of distribution of trial perturbation with finite inverse moments and vector-function $\mathcal{K}_n(\cdot)$, defined by rule:

$$\mathcal{K}_n(\Delta_n) = \begin{pmatrix} \frac{1}{\Delta_n^{(1)}} \\ \frac{1}{\Delta_n^{(2)}} \\ \vdots \\ \frac{1}{\Delta_n^{(q)}} \end{pmatrix}.$$

With same vector-funcrtion $\mathcal{K}_n(\cdot)$ and constraints on distribution of trial simultaneous perturbation H.-F. Chen and others in paper [14] was proposed to use algorithm(6).

We will use instead of algorithm (4) slightly different one with projection when we formulate the main result:

$$\begin{cases} x_n = \hat{\theta}_{n-1} + \beta_n \Delta_n,\ y_n = F(x_n, w_n) + v_n\ , \\ \hat{\theta}_n = \mathcal{P}_{\Theta_n}(\hat{\theta}_{n-1} - \frac{\alpha_n}{\beta_n}\mathcal{K}_n(\Delta_n) y_n), \end{cases} \tag{8}$$

for which it is more comfortable to prove. In this algorithm $\mathcal{P}_{\Theta_n}(\cdot)$ are projecting operators on some convex closed bounded subsets $\Theta_n \subset \mathbb{R}^q$, which contain, starting from some $n \geq 1$, the answer point θ. If the bounded closed convex set Θ: $\theta \in \Theta$ is known, then we can decide that $\Theta_n = \Theta$. In other case sets $\{\Theta_n\}$ can be wider each time up to infinity.

Some specifics of the task can allow to construct decreasing sequence $\{\Theta_n\}$.

3 Main conditions

Consider $\rho \in (1, 2]$. We will use following notation: $\mathrm{E}\{\cdot\}$ — for expectation; $\|\cdot\|$, $\|\cdot\|_\rho$ and $(\cdot,\cdot)$ — for Euclidean norm, norm in l_ρ space and scalar product in $\mathbb{R}^q$; $\mathcal{F}_{n-1}$ — is for σ-algebra of probabilistic events, derived from random values $\hat{\theta}_0, \hat{\theta}_1, \ldots, \hat{\theta}_{n-1}$, constructed by algorithm (5) (or (6), or (8)); using algorithms (5) or (6)

$$\bar{w}_n = \begin{pmatrix} w_{2n} \\ w_{2n-1} \end{pmatrix},\ \ \bar{v}_n = \kappa(v_{2n} - v_{2n-1}),$$

$$\kappa = \begin{cases} \frac{1}{2}, \text{for (5)}, \\ 1,\ \text{for (6)}, \end{cases}$$

$$F_w = \max_{x \in \mathbb{R}^q} E_{w'}\{E_{w''}\{\kappa^\rho |F(x, w') - F(x, w'')|^\rho\}\},$$

and when constructing estimates by algorithm (8)

$$\bar{v}_n = v_n, \bar{w}_n = w_n, F_w = \mathrm{E}_w\{|F(\theta, w)|^\rho\}.$$

Consider a function

$$V(x) = \|x - \theta\|_\rho^\rho = \sum_{i=1}^q |x^{(i)} - \theta^{(i)}|^\rho,$$

where θ — is an optimal vector which we need to find.

Let's formulate main assumptions.

(A) Function $f(x)$ has a unique minimum and

$$(\nabla V(x), \nabla f(x)) \geq \mu V(x),\ \ \forall x \in \mathbb{R}^q$$

with some constant $\mu > 0$.

(B) $\forall w$ gradients of functions $F(\cdot, w)$ satisfy the condition

$$\|\nabla_x F(x, w) - \nabla_x F(y, w)\|_\rho \leq M\|x - y\|_\rho,\ \forall x, y \in \mathbb{R}^q$$

with some constant $M > 0$.

(C) Local condition of Lebesgue for $\nabla_x F(x, \cdot)$: $\forall x\ \exists$ neighbourhood U_x : $\forall x' \in U_x\ \exists$ function $\Phi_x(\cdot) : \mathbb{R}^p \to \mathbb{R}$, $\mathrm{E}_w\{\Phi_x(w)\} < \infty$:

$$|\nabla_x F(x', w)| \leq \Phi_x(w)\ \text{ for almost all }\ w.$$

(D) For $\mathcal{K}_n(\cdot)$ and $\mathrm{P}_n(\cdot)$, $n = 1, 2, \ldots$ conditions are satisfied:

$$\bar{K} = F_w \sup_{n=1,2,\ldots} \int \|\mathcal{K}_n(x)\|_\rho^\rho \mathrm{P}_n(dx) < \infty,$$

$$\tilde{K} = \sup_{n=1,2,\ldots} \int \|\mathcal{K}_n(x)\|_\rho \|x\|_\rho \|x\|_{\frac{\rho}{\rho-1}} \mathrm{P}_n(dx) < \infty.$$

(E) For every $n \geq 1$

$$\xi_n = \|\mathrm{E}\{\mathcal{K}_n(\Delta_n)\bar{v}_n|\mathcal{F}_{n-1}\}\|_\rho^\rho \leq C_{\Delta v}\beta_n^2,$$

$$\mathrm{E}\{\|\mathcal{K}_n(\Delta_n)\bar{v}_n\|_\rho^\rho\} \leq \sigma_n^\rho.$$

In case of $\rho = 2$ conditions **(A)** and **(B)** have the same form as it was in earlier papers (for example, [10]):

(A') — function $f(\cdot)$ is strictly convex, that is

$$\langle x - \theta, \nabla f(x)\rangle \geq \mu\|x - \theta\|^2,\ \ \forall x \in \mathbb{R}^q.$$

(B')— Lipschitz condition for gradients of functions $F(\cdot, w)$: $\forall x, \theta \in \mathbb{R}^q$

$$\|\nabla_x F(x, w) - \nabla_x F(y, w)\| \leq M\|x - \theta\|.$$

4 Convergence of the sequence of estimates

Denote:

$$\begin{aligned}
\nu_n &= 2\rho\alpha_n^\rho \beta_n^{-\rho}, \\
\gamma_n &= \alpha_n \rho\mu - \alpha_n(\beta_n c(\rho - 1) + \delta_n M^\rho) \\
\phi_n &= \alpha_n \beta_n c + 2^{\rho-1}\bar{K}\nu_n + \chi_n, c = M\tilde{K} + C_{\Delta v}, \\
\chi_n &= \begin{cases} 2^{1-\rho}\eta_n + \psi_n, \text{for}(5), (6) \\ \eta_n + 2^{1-\rho}\psi_n, \text{for}(8) \end{cases} \\
\mathrm{K}(x) &= 2^{3\rho-2}\begin{cases} \|x\|_{\frac{\rho}{\rho-1}}^\rho & \text{for}(5), (6) \\ (\frac{\mathrm{diam}(\Theta_n)}{\beta_n} + \|x\|_{\frac{\rho}{\rho-1}})^\rho, & \text{for}(8) \end{cases} \\
\psi_n &= \alpha_n \delta_n \mathrm{E}_w\{\|\nabla_x F(\theta, w)\|_\rho^\rho\}, \\
\delta_n &= \alpha_n^{\rho-1}\rho \int \|\mathcal{K}_n(x)\|_\rho^\rho \mathrm{K}(x)\mathrm{P}_n(dx), \\
\eta_n &= \rho\alpha_n^\rho \beta_n^\rho M^\rho \int \|\mathcal{K}_n(x)\|_\rho^\rho \|x\|_\rho^\rho \mathrm{K}(x)\mathrm{P}_n(dx),
\end{aligned}$$

$\mathrm{diam}(\cdot)$ — Euclidean diameter of a set in metrics $l_{\frac{\rho}{\rho-1}}$.

Theorem 1. *Let be $\rho \in (1,2]$ and the conditions are satisfied:*
(A–C); **(7)**; *random values $\{\bar{v}_k, \bar{w}_k, \Delta_k\}_{k=1}^{n-1}$ do not depend on $\bar{w}_n$ è Δ_n, and random vector w_n does not depend on Δ_n;*
$\forall n,\ 0 \le \gamma_n \le 1,\ \sum_n \gamma_n = \infty,\ \mu_n \to 0$ with $n \to \infty$, where

$$\mu_n = \frac{\phi_n + \nu_n \sigma_n^\rho}{\gamma_n}, \quad z_n = \left(1 - \frac{\mu_{n+1}}{\mu_n}\right) \frac{1}{\gamma_{n+1}}.$$

Then: 1) *sequence of estimates $\{\hat{\theta}_n\}$, given by algorithm* (8) *(or* (5)*, or* (6)*), converges to a point θ in following sense: $\mathrm{E}\{V(\hat{\theta}_n)\} \to 0$ when $n \to \infty$;*
2) *if $\overline{\lim}_{n\to\infty}\ z_n \ge z > 1$,* **then** $\mathrm{E}\{V(\hat{\theta}_n)\} = \mathcal{O}\left(\prod_{i=0}^{n-1}(1-\gamma_i)\right)$;
3) *if $z_n \ge z > 1\ \forall n$,* **then** $\mathrm{E}\{V(\hat{\theta}_n)\} \le (\mathrm{E}\{V(\hat{\theta}_0)\} + \frac{\mu_0}{z-1}) \prod_{i=0}^{n-1}(1-\gamma_i)$;
4) *if, moreover, $\sum_n \phi_n + \nu_n \mathrm{E}\{\|\mathcal{K}_n(\Delta_n)\bar{v}_n\|_\rho^\rho | \mathcal{F}_{n-1}\} < \infty$ a. s.,*
then *$\hat{\theta}_n \to \theta$ while $n \to \infty$ a. s. and*

$$\mathrm{P}\{\forall n \ge n_0\, V(\hat{\theta}_n) \le \varepsilon\} \ge$$

$$\ge 1 - \frac{\mathrm{E}\{V(\hat{\theta}_{n_0})\} + \sum_{n=n_0}^\infty \phi_n + \nu_n \sigma_n^\rho}{\varepsilon}.$$

Proof of the theorem 1 can be found in the last section.

Note 1. For function $F(x,w) = wf(x)$ conditions **(A)**–**(C)** of the theorem 1 are satisfied, if function $f(x)$ satisfies the conditions **(A)** è **(B)**.

Note 2. In [3, 4] are formulated close results about accuracy of estimation and speed of convergence of algorithms (8) and (5).

Note 3. The problem of estimation of parameters in linear regression model with observations (3) when $\theta_n = \theta$ corresponds to minimization of a functional of average risk

$$f(x) = \frac{1}{2}(x-\theta)^{\mathrm{T}}(x-\theta).$$

Note 4. In the theorem 1 noise v_n in observations can be called *almost arbitrary*, because it may be not random (determined), but unknown and bounded, or be a realisation of some stochastic process with arbitrary structure of dependencies. In particular, for proving the the statements of the theorem 1 there is no need to assume anything about dependencies between $\bar{v}_n$ and $\mathcal{F}_{n-1}$.

Note 5. Although algorithms (5) and (6) seem to be similar,in case of arbitrary noise in observations the use of the second in real time systems is better.

For algorithm (5) satisfaction of the condition about independency of the noise v_{2n} from trial perturbation Δ_n is quite strict, because at the moment $2n-1$ vector Δ_n has been already used in the system. Using the algorithm (6) noise v_{2n} and vector of trial perturbation Δ_n enter the system simultaneously, what allows to hope on their independency.

Note 6. For another generalisation of conditions of convergence for the algorithms (5), (6) and (8) sequences $\{\alpha_n\}$ and $\{\beta_n\}$ can be random, measurable relatively σ-algebra $\mathcal{F}_n$. Practical need in such deneralisation appear, for instance, when, in parallel with computation of estimates by SPSA algorithm additional conditions of the task give information about the quality of estimation. If estimates are "bad", then it is possible to make the speed of convergence of sequence $\{\alpha_n\}$ to zero lower, maybe make the values of the sequence bigger for a while. **Note 7** The generalisation about noise properties, namely existence of $\rho \in (1,2]$-moments for w, clarifies that SPSA-class algorithms can be used in almost-heavy-tailed noise case. The middle value of three independent identically distributed heavy-tailed random variables satisfies heavy tail property.

5 Proof of the Theorem 1

We denote for the algorithm (8): $\bar{y}_n = y_n$, for (5): $\bar{y}_n = (y_{2n} - y_{2n-1})/2$, for (6): $\bar{y}_n = y_{2n} - y_{2n-1}$.

For estimates of the algorithm (8) applying the projector properties we get $V(\hat{\theta}_n) = V(\mathcal{P}_{\Theta_n}(\hat{\theta}_{n-1} - \frac{\alpha_n}{\beta_n}\mathcal{K}_n(\Delta_n)\bar{y}_n)) \le V(\hat{\theta}_{n-1} - \frac{\alpha_n}{\beta_n}\mathcal{K}_n(\Delta_n)\bar{y}_n)$. For other algorithms' estimates we get equality. Using properties of chosen function $V(x)$, from middle-value theorem with some $t \in (0,1)$ we sequentially derive:

$$V(\hat{\theta}_n) \le V(\hat{\theta}_{n-1}) - \frac{\alpha_n}{\beta_n}(\nabla V(\hat{\theta}_{mid}), \mathcal{K}_n(\Delta_n)\bar{y}_n) =$$

$$V(\hat{\theta}_{n-1}) - \frac{\alpha_n}{\beta_n}(\nabla V(\hat{\theta}_{n-1} - t\frac{\alpha_n}{\beta_n}\mathcal{K}_n(\Delta_n)y_n),$$

$$\mathcal{K}_n(\Delta_n)\bar{y}_n) = V(\hat{\theta}_{n-1}) - \rho\frac{\alpha_n}{\beta_n}\sum_{i=1}^q \left|\hat{\theta}_{n-1}^{(i)} - \theta^{(i)} - \right.$$

$$\left. -t\frac{\alpha_n}{\beta_n}\mathcal{K}_n(\Delta_n)^{(i)}\bar{y}_n\right|^{\rho-1} \mathrm{sign}_n^{(i)}(t)\mathcal{K}_n(\Delta_n)^{(i)}\bar{y}_n,$$

where $\mathrm{sign}_n^{(i)}(t) = 0$ or ± 1 depending on the sign of expression $\hat{\theta}_{n-1}^{(i)} - \theta^{(i)} - t\frac{\alpha_n}{\beta_n}\mathcal{K}_n(\Delta_n)^{(i)}\bar{y}_n$. Denote $\widetilde{\mathrm{sign}}_{n-1}^{(i)} = 0$ or ± 1 depending on the sign of $\hat{\theta}_{n-1}^{(i)} - \theta^{(i)}$. Next, using the inequality

$$-\mathrm{sign}(c-d)|c-d|^{\rho-1}b \le -\mathrm{sign}(c)|c|^{\rho-1}b + 2^{2-\rho}|d|^{\rho-1}|b|$$

for all $b, c, d \in \mathbb{R}$, we get:

$$V(\hat{\theta}_n) \le V(\hat{\theta}_{n-1}) - \rho\frac{\alpha_n}{\beta_n}\sum_{i=1}^q \left|\hat{\theta}_{n-1}^{(i)} - \theta^{(i)}\right|^{\rho-1}.$$

$$\cdot \widetilde{\mathrm{sign}}_{n-1}^{(i)} \mathcal{K}_n(\Delta_n)^{(i)} \bar{y}_n + 2^{2-\rho} \rho \frac{\alpha_n}{\beta_n} \cdot$$

$$\cdot \sum_{i=1}^{q} \left| t \frac{\alpha_n}{\beta_n} \mathcal{K}_n(\Delta_n)^{(i)} \bar{y}_n \right|^{\rho-1} |\mathcal{K}_n(\Delta_n)^{(i)} \bar{y}_n| \leq$$

$$\leq V(\hat{\theta}_{n-1}) - \frac{\alpha_n}{\beta_n} \sum_{i=1}^{q} \nabla V(\hat{\theta}_{n-1})^{(i)} \mathcal{K}_n(\Delta_n)^{(i)} \bar{y}_n +$$

$$+ 2^{2-\rho} \rho \left(\frac{\alpha_n}{\beta_n} \right)^{\rho} \sum_{i=1}^{q} |\mathcal{K}_n(\Delta_n)^{(i)} \bar{y}_n|^{\rho}.$$

Consequently, we get:

$$V(\hat{\theta}_n) \leq V(\hat{\theta}_{n-1}) - \frac{\alpha_n}{\beta_n} (\nabla V(\hat{\theta}_{n-1}), \mathcal{K}_n(\Delta_n) \bar{y}_n) + \\ + 2^{2-\rho} \nu_n \|\mathcal{K}_n(\Delta_n) \bar{y}_n\|_{\rho}^{\rho}. \quad (9)$$

From the model of observations (1), conciderring middle-value theorem for the function $F(\cdot, w_n)$, we derive with some $t', t'' \in (0, 1)$ next formula:

$$\bar{y}_n = \bar{F}_n(0,0) + \bar{F}_n'(t', t'') + \bar{v}_n,$$

where for algorithms (8), (5) and (6) we denote:

$$\bar{F}_n(t', t'') = \begin{cases} F(\hat{\theta}_{n-1} + t'\beta_n \Delta_n, w_n), (8) \\ \frac{1}{2}(F(\hat{\theta}_{n-1} + t'\beta_n \Delta_n, w_{2n}) - \\ - F(\hat{\theta}_{n-1} - t''\beta_n \Delta_n, w_{2n-1})), (5) \\ F(\hat{\theta}_{n-1} + t'\beta_n \Delta_n, w_{2n}) - \\ - F(\hat{\theta}_{n-1}, w_{2n-1}), (6) \end{cases}$$

and

$$\bar{F}_n'(t', t'') = \frac{\partial \bar{F}_n(t', t'')}{\partial t'} + \frac{\partial \bar{F}_n(t', t'')}{\partial t''}.$$

Let's use expectation operation relatively σ-algebra $\mathcal{F}_{n-1}$.

From independence $\mathcal{K}_n(\Delta_n)$ from $\bar{w}_n$ and symmetry of distribution $\mathrm{P}_n(\cdot)$ (condition (7)) we get

$$\mathrm{E}\{\mathcal{K}_n(\Delta_n) \bar{F}_n(0,0) | \mathcal{F}_{n-1}\} = 0.$$

Consequently, for conditional expectation of second term in formula (9) we sequentially get

$$-\frac{\alpha_n}{\beta_n} \mathrm{E}\{(\nabla V(\hat{\theta}_{n-1}), \mathcal{K}_n(\Delta_n) \bar{y}_n) | \mathcal{F}_{n-1}\} \leq$$

$$\leq -\frac{\alpha_n}{\beta_n} (\nabla V(\hat{\theta}_{n-1}), \mathrm{E}\{\mathcal{K}_n(\Delta_n) \bar{F}_n'(t', t'') | \mathcal{F}_{n-1}\}) + \quad (10)$$

$$+ \frac{\alpha_n}{\beta_n} |(\nabla V(\hat{\theta}_{n-1}), \mathrm{E}\{\mathcal{K}_n(\Delta_n) \bar{v}_n | \mathcal{F}_{n-1}\})|.$$

Using Hölder inequality [19] (p. 129), Jensen [19] (p. 210), Yung [20] (p. 280): $a^{1/r} b^{1/s} \leq \frac{1}{r} a + \frac{1}{s} b$, $r > 1$, $a, b > 0$, $\frac{1}{r} + \frac{1}{s} = 1$, and condition **(E)**, for the last term we get an upper bound

$$\frac{\alpha_n}{\beta_n} |(\nabla V(\hat{\theta}_{n-1}), \mathrm{E}\{\mathcal{K}_n(\Delta_n) \bar{v}_n | \mathcal{F}_{n-1}\})| \leq \quad (11)$$

$$\rho \frac{\alpha_n}{\beta_n} V(\hat{\theta}_{n-1})^{\frac{\rho-1}{\rho}} \times \|\mathrm{E}\{\mathcal{K}_n(\Delta_n) \bar{v}_n | \mathcal{F}_{n-1}\}\|_{\rho} \leq$$

$$\leq \alpha_n \beta_n C_{\Delta v} \left((\rho - 1) V(\hat{\theta}_{n-1}) + 1 \right).$$

Using the independence of $\bar{w}_n$ and Δ_n, local Lebesgue condition **(C)** for $\nabla_x F(\hat{\theta}_{n-1}, \cdot)$ and condition (7), also get

$\nabla f(\hat{\theta}_{n-1}) = \mathrm{E}\{\nabla_x F(\hat{\theta}_{n-1}, w) | \mathcal{F}_{n-1}\} = \beta_n^{-1} \mathrm{E}\{\mathcal{K}_n(\Delta_n) \bar{F}_n'(0,0) | \mathcal{F}_{n-1}\}$.

Denote the difference $\tilde{F}_n' = \bar{F}_n'(t', t'') - \bar{F}_n'(0,0)$ and estimate it's abcolute value. Considering condition **(B)** and the fact that $t' \in (0,1)$, for the algorithm (8) derive $|\tilde{F}_n'| = |(\nabla_x F(\hat{\theta}_{n-1} + t'\beta_n \Delta_n, w_n) - \nabla_x F(\hat{\theta}_{n-1}, w_n), \beta_n \Delta_n)| \leq$

$$\leq \beta_n \|\Delta_n\|_{\frac{\rho}{\rho-1}} \|\nabla_x F(\hat{\theta}_{n-1} + t'\beta_n \Delta_n, w_n) - \\ - \nabla_x F(\hat{\theta}_{n-1}, w_n)\|_{\rho} \leq M \beta_n^2 \|\Delta_n\|_{\rho} \|\Delta_n\|_{\frac{\rho}{\rho-1}}.$$

For algorithms (5) and (6) the same formula can be derived analogously.

From the last formula, bounding the scalar product in the first term of (10), from conditions **(A)**—**(D)** and inequalities of Hölder, Jensen, Yung we get

$$(\nabla V(\hat{\theta}_{n-1}), \mathrm{E}\{\mathcal{K}_n(\Delta_n) \bar{F}_n'(t', t'') | \mathcal{F}_{n-1}\}) =$$

$$\beta_n (\nabla V(\hat{\theta}_{n-1}), \nabla f(\hat{\theta}_{n-1})) + (\nabla V(\hat{\theta}_{n-1}),$$

$$\mathrm{E}\{\mathcal{K}_n(\Delta_n) \tilde{F}_n' | \mathcal{F}_{n-1}\}) \geq \beta_n \mu V(\hat{\theta}_{n-1}) -$$

$$- \rho V(\hat{\theta}_{n-1})^{\frac{\rho-1}{\rho}} \mathrm{E}\{\|\mathcal{K}_n(\Delta_n)\|_{\rho} |\tilde{F}_n'| \; | \mathcal{F}_{n-1}\}) \geq$$

$$\geq \beta_n \mu V(\hat{\theta}_{n-1}) - M \beta_n^2 \rho \left(\frac{\rho - 1}{\rho} V(\hat{\theta}_{n-1}) + \frac{1}{\rho} \right)$$

$$\mathrm{E}\{\|\mathcal{K}_n(\Delta_n)\|_{\rho} \|\Delta_n\|_{\rho} \|\Delta_n\|_{\frac{\rho}{\rho-1}}\} \geq$$

$$\geq \beta_n \mu V(\hat{\theta}_{n-1}) - \beta_n^2 M \tilde{K} \left((\rho - 1) V(\hat{\theta}_{n-1}) + 1 \right) \quad (12)$$

Then, from (11) and (12) for conditional expectation of the second term in formula (9), to continue (10), get,

$$\cdots \leq -\alpha_n \rho \mu V(\hat{\theta}_{n-1}) + \alpha_n \beta_n (M \tilde{K} + C_{\Delta v}) \cdot \\ \cdot \left((\rho - 1) V(\hat{\theta}_{n-1}) + 1 \right). \quad (13)$$

Let's bound the conditional expectation of the third term in right side of inequality (9). Using Jensen inequality $(\frac{a+b}{2})^{\rho} \leq \frac{1}{2}(a^{\rho} + b^{\rho})$ for convex function x^{ρ}, we get

$$2^{1-\rho} \nu_n \mathrm{E}\{\|\mathcal{K}_n(\Delta_n) \bar{y}_n\|_{\rho}^{\rho} | \mathcal{F}_{n-1}\} \leq \nu_n \mathrm{E}\{\|\mathcal{K}_n(\Delta_n) \cdot \\ \cdot \bar{F}_n(1,1)\|_{\rho}^{\rho} | \mathcal{F}_{n-1}\} + \nu_n \mathrm{E}\{\|\mathcal{K}_n(\Delta_n) \bar{v}_n\|_{\rho}^{\rho} | \mathcal{F}_{n-1}\}. \quad (14)$$

For algorithms (5) and (6) we get $|\bar{F}_n(1,1)|^{\rho} \leq 2^{\rho-1}(|\bar{F}_n(0,0)|^{\rho} + |\bar{F}_n'(t', t'')|^{\rho}) \leq 2^{\rho-1} |\bar{F}_n(0,0)|^{\rho} + 2^{\rho-1} |\bar{F}_n'(t', t'') + F_{\theta} - F_{\theta}|^{\rho} \leq 2^{\rho-1} |\bar{F}_n(0,0)|^{\rho} + 2^{2\rho-2}(|F_{\theta}|^{\rho} + |\bar{F}_n'(t', t'') - F_{\theta}|^{\rho})$ using

$$F_{\theta} = \begin{cases} \frac{1}{2}\big((\nabla F(\theta, w_{2n}) + \nabla F(\theta, w_{2n-1})), \beta_n \Delta_n\big), \text{for (5)}, \\ \big(\nabla F(\theta, w_{2n}), \beta_n \Delta_n\big), \text{ for (6)}, \end{cases}$$

From this, using Hölder inequality [19] (p. 129) and conditions **(B)**, **(D)**, continue (14) $\cdots \leq \alpha_n \delta_n (V(\hat{\theta}_{n-1}) + \mathrm{E}_w\{\|\nabla_x F(\theta, w)\|_{\rho}^{\rho}\}) + 2^{2\rho-2} \hat{K} M^{\rho} \nu_n \beta_n^{2\rho} + 2^{\rho-1} \bar{K} \nu_n + \nu_n \xi_n \leq$

$$\leq \alpha_n \delta_n M^{\rho} V(\hat{\theta}_{n-1}) + 2^{\rho-1} \bar{K} \nu_n + \chi_n + \nu_n \xi_n. \quad (15)$$

In case of the algorithm (8) for some point x_m, which belongs to a segment between $\hat{\theta}_{n-1} + \beta_n \Delta_n$ and θ, from the middle-value theorem and Jensen inequality we get:

$$|\bar{F}_n(1,1)|^{\rho} = |F(\theta, w_n) + (\nabla_x F(x_m, w_n), \hat{\theta}_{n-1} +$$

$$\beta_n\Delta_n - \theta)|^\rho \le 2^{\rho-1}|F(\theta,w_n)|^\rho +$$
$$+2^{2\rho-2}(\|\nabla_x F(x_m,w_n) - \nabla_x F(\theta,w_n)\|^\rho_\rho +$$
$$+\|\nabla_x F(\theta,w_n)\|^\rho_\rho)(\|\hat\theta_{n-1}-\theta\|_{\frac{\rho}{\rho-1}} +$$
$$+\|\beta_n\Delta_n\|_{\frac{\rho}{\rho-1}})^\rho \le 2^{\rho-1}|F(\theta,w_n)|^\rho +$$
$$+2^{2\rho-2}\Big(2^{\rho-1}M^\rho\big(V(\hat\theta_{n-1}) + \beta_n^\rho\|\Delta_n\|^\rho_\rho\big) +$$
$$+\|\nabla_x F(\theta,w_n)\|^\rho_\rho\Big)\beta_n^\rho \mathrm{K}(\Delta_n).$$

From last inequality for the algorithm (8) we get the same inequality (15).

Using the discussed notation and the bounds got above (13) and (15), inequalities(9) we can change on

$$V(\hat\theta_n) \le (1-\gamma_n)V(\hat\theta_{n-1}) + \phi_n + \nu_n\xi_n.$$

Using the unconditional expectation from left and right sides of the last inequality, we get inequalities

$$\mathrm{E}\{V(\hat\theta_n)\} \le (1-\gamma_n)\mathrm{E}\{V(\hat\theta_{n-1})\} + \phi_n + \nu_n\sigma_n^\rho,$$

from which the statements of the theorem 1 can be easy derived from corresponding [12] statements of the theorem 1.

Proof of the theorem 1 finished.

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