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Algorithm for Stochastic Approximation with Trial Input Perturbation in the Nonstationary Problem of Optimization

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Abstract—Consideration was given to the randomized stochastic approximation algorithm with simultaneous trial input perturbation and two measurements used to optimize the unconstrained nonstationary functional. The upper boundary of the mean-square residual was established under conditions of single differentiability of the functional and almost arbitrary noise. Efficiency of the algorithm was illustrated by an example of stabilization of the resulting estimates for the multidimensional case under dependent observation noise.

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1. INTRODUCTION

Many practical applications need to optimize one or another mean risk functional. Although sometimes the extremal values can be established analytically, the engineering systems often deal with an unknown functional whose value or gradient can be calculated at the given points. One also encounters problems where the optimized functional may vary in time and its point of extremum may drift. In such cases, the problems may be posed differently depending on the aims of optimization and the measurable information. Usually consideration is given to two variants of behavior of the drift of functional's point of minimum where some asymptotic functional to which other functionals converge with time either exists or not [1]. The present paper considers only the more complicated, latter case, the former case yielding to the approaches developed in detail in [2].

The problems of optimization may be considered in discrete and continuous times. The present authors confine themselves to the models of the first type. Let $f(x, n)$ be a functional minimized at the time instant n ($n \in \mathbf{N}$). The Newton and gradient methods that are applicable in the case of twice differentiable functional, provided that $r < \nabla_x^2 f(x, n) < R$, were discussed in detail in [3] for solution of such problems. Both methods imply that it is possible to measure directly the functional's gradient at an arbitrary point.

Physical measurements always imply presence of noise. The theoretically substantiated algorithms sometimes fail to give in practice consistent estimates of the extremum point. Algorithm's noise immunity is a must in actually all engineering applications. The Robbins–Monro and Kiefer–Wolfowitz methods of stochastic approximation were developed in the 1950's for solution of problems in the conditions of noise. A detailed account of their history is given in [2–4]. The general approach to determining the extremum that is used in the stochastic approximation algorithms may be formalized as follows:

$$\hat{\theta}_{n+1} = \hat{\theta}_n - \alpha_n \hat{g}_n(\hat{\theta}_n), \tag{1}$$

where $\{\hat{\theta}_n\}$ is the sequence of estimates of the point of extremum as generated by the algorithm and g_n is the pseudogradient (replacing the gradient from the method of Newton) which in the average must coincide with the gradient and be close to zero if the argument tends to the point extremum. Simplicity and recurrence are important properties of the algorithms like (1) which accounts for their popularity in diverse areas of science and technology.

The stochastic approximation algorithms with simultaneous trial input perturbation and one or two noisy measurements of the minimized function at each iteration appeared in the publications of various researchers in the late 1980s and early 1990s [5–9]. In the English literature they were christened *simultaneous perturbation stochastic approximation* (SPSA) algorithms [8]. For the almost arbitrary observation noise, these algorithms are known to have consistent estimates which must be bounded only in a certain sense and independent at each iteration with a random trial input perturbation [2]. Moreover, at each iteration the function under consideration is measured only once or twice independently of the dimension of the state space. At that, the algorithms have an optimal order of the asymptotic convergence rate (in terms of the number of iterations) [7], which means that in the multidimensional case ($d \gg 1$) the convergence rate is much higher as compared with the classical algorithms estimating the gradient in terms of the finite difference with the number of measurements at each step being directly proportional to d.

Initially the algorithms of stochastic approximation were substantiated in the conditions of minimization of the stationary functionals. A version of the gradient descent algorithm for the nonstationary case was presented in [3] where its consistency in a sense was proved. This idea was further developed for a more general case [1]. It was proposed in [2] to optimize the nonstationary functional using the stochastic approximation algorithms with simultaneous trial input perturbation that could be more efficient because at each step they rely only on one or two measurements and, consequently, can faster adapt to the functional's variations. Additionally, these algorithms, as was already noticed, are more noise-immune. This idea was clearly formalized in [10] where preliminary precise results concerning its applicability were obtained.

In the present paper we consider application of the stochastic approximation algorithm with the simultaneous trial input perturbation to optimization of the nonstationary functional. In Section 2 we formulate in detail the optimization problem which is essentially more general than that of [1, 3]: the minimized functional is differentiable once and not twice, direct measurement of the gradient is not assumed, and observation noise can be almost arbitrary. The algorithm is formulated in Section 3, and the assertion of stabilizability of the mean-square estimate residual will be formulated in Section 4 and proved in the Appendix. Efficiency of the algorithm is illustrated in Section 5 by way of example of stabilization of the resulting estimates in the multidimensional case of tracking a point drifting in space under dependent observation noise.

2. FORMULATION OF THE PROBLEM

Let time be discrete and defined by the number of the step (iteration) $n = 0, 1, \ldots$, $\{F(\cdot, \cdot, n) : \mathbf{R}^d \times \mathbf{R}^p \to \mathbf{R}\}$ be a set of functions of two vector variables, and it be possible to measure at each step n the value

$$y_n = F(x_n, w_n, n) + v_n, \tag{2}$$

where x_n are certain (selected) points of observation (experiment design), w_n are independent random variables that express uncontrollable uncertainty, are defined over some probabilistic space Ω, and have identical, generally speaking, unknown distribution $\mathrm{P_w}(\cdot)$, and v_n are observation distortions (also may be nonrandom).

Let us consider the problem of minimization of the nonstationary mean-risk functional

$$f(x, n) = E_w F(x, w, n) = \int\limits_{\mathbf{R}^p} F(x, w, n) \mathrm{P_w}(dw) \to \min_x . \tag{3}$$

Needed is to estimate the point of minimum θ_n of the function $f(x,n)$ which varies in time:

$$\theta_n = \text{argmin}_x f(x,n).$$

We introduce two definitions characterizing behavior of the estimates of the minimum points of the nonstationary functional (3).

Definition 1. The sequence of estimates $\hat{\theta}_n$ of the minimum points θ_n *is mean-square stabilizable* if there exists $C > 0$ such that

$$E\left\|\hat{\theta}_n - \theta_n\right\|^2 \le C\ \forall n,$$

here and below E stands for the expectation.

Definition 2. The number $\bar{L}$ is called the *asymptotic boundary of the mean-square residues of estimation* if

$$\overline{\lim_{n\to\infty}}\left(E\left\|\hat{\theta}_n - \theta_n\right\|^2\right)^{\frac{1}{2}} \le \bar{L} < \infty$$

is satisfied for the sequence of estimates $\{\hat{\theta}_n\}$ of the minimum points θ_n.

Below we consider the problem of constructing the sequence of stabilizing estimates $\{\hat{\theta}_n\}$ under the following conditions.

(**A**) The functions $f(\cdot,n)$ are strongly convex in the first argument for each n:

$$\langle \nabla f(x,n), x - \theta_n\rangle \ge \mu\|x - \theta_n\|^2.$$

(**B**) The gradient $\nabla F(\cdot,w,n)$ satisfies $\forall n$ and $\forall w$ the Lipschitz condition with the constant M:

$$\|\nabla F(x,w,n) - \nabla F(y,w,n)\| \le M\|x-y\|,$$

and the gradient has the following property.

(**C**) Local Lebesgue property: $\forall x \in \mathbb{R}^d$ and $\forall n$ $\exists$ the neighborhood $U_{x,n}$ and the function $\Phi_x(w,n) : E_w\Phi_x(w,n) < \infty$ such that $\forall x' \in U_{x,n}$ $\|\nabla F(x,w,n)\| \le \Phi_x(w,n)$.

(**D**) The drift speed of the minimum point is bounded by the conditions

$$\forall n\ \ \|\theta_n - \theta_{n-1}\| \le A,\ \ E_w|\nabla F(\theta_n,w,n)|^2 \le B,$$
$$E_{w_1,w_2}|F(x,w_1,n) - F(x,w_2,n-1)|^2 \le C\|x - \theta_{n-2}\|^2 + D.$$

(**E**) For the observation noise v_n, satisfied are the conditions

$$|v_{2n} - v_{2n-1}| \le \sigma_v$$

or

$$E\{|v_{2n} - v_{2n-1}|^2\} \le \sigma_v^2$$

if the noise is a sequence of random variables.

We notice that this condition is satisfied by the deterministic but bounded sequences $\{v_n\}$.

Condition (**C**) enables one to permute the operations of integration and differentiation at substantiating stabilizability of the estimates.

Constraints like (**D**) include both drift of the random-walk type and the drift in a certain direction. For example, it is possible to consider the constraint

$$\theta_n = \theta_{n-1} + A + \xi_n,$$

where ξ_n is a bounded random variable. Such a condition is assumed also in [3], and in [1] is it relaxed a little. The mean-square stabilizability of the estimates of the algorithm for determining the minimum under conditions (**D**) means that it can be applied to a wide class of problems.

3. ESTIMATION ALGORITHM

Let the sequence $\{\Delta_n\}$ of simultaneous trial perturbations that are input in the estimation algorithm be a realization of a sequence of independent Bernoulli random vectors from $\mathbb{R}^d$ with each component independently assuming values $\pm\frac{1}{\sqrt{d}}$ with the probabilities $\frac{1}{2}$.

We take an initial vector $\hat{\theta}_0 \in \mathbb{R}^d$ and estimate the sequence of the minimum points $\{\theta_n\}$ by the sequence $\{\hat{\theta}_n\}$, defined by the following algorithm of stochastic optimization with simultaneous trial input perturbation:

$$\begin{cases} x_{2n} = \hat{\theta}_{2n-2} + \beta\Delta_n, \quad x_{2n-1} = \hat{\theta}_{2n-2} - \beta\Delta_n \\ y_n = F(x_n, w_n, n) + v_n \\ \hat{\theta}_{2n} = \hat{\theta}_{2n-2} - \dfrac{\alpha}{2\beta}\Delta_n(y_{2n} - y_{2n-1}) \\ \hat{\theta}_{2n-1} = \hat{\theta}_{2n-2}. \end{cases} \tag{4}$$

To substantiate the mean-square stabilization of the estimates of algorithm (4), we assume that

(**F**) The random variables Δ_n (randomization of the algorithm's input) and w_{2n} and w_{2n-1} are mutually independent, independent of $\hat{\theta}_0$, w_k, $k = 1, 2, \ldots, 2n-2$, and of θ_k, $k = 1, 2, \ldots, 2n$. If it is assumed that v_k are of random nature, then Δ_n, w_{2n}, and w_{2n-1} are independent of v_k, $k = 1, 2, \ldots, 2n$.

4. MEAN-SQUARE STABILIZATION OF ESTIMATES

We denote $H = A + \alpha\beta M$.

Theorem 1. *Let Conditions* (**A**)–(**F**) *be satisfied and the parameters of the algorithm* α, β *be selected so that*

$$K = 2\alpha\mu - 3(C + 6M^2\beta^2)\frac{\alpha^2}{\beta^2} > 0.$$

If $E\|\theta_0 - \hat{\theta}_0\|^2 < \infty$, *then the estimates of algorithm* (4) *are mean-square stabilized.*

Additionally, $\forall\, n$ *and any parameter* $\delta > 0$ *satisfying the condition*

$$\bar{K} = 1 - K + \delta H < 1,$$

valid are the estimates

$$E\left\|\theta_n - \hat{\theta}_n\right\|^2 \le \bar{K}^n E\left\|\theta_0 - \hat{\theta}_0\right\|^2 + \frac{(L + H/\delta)(1 - \bar{K}^n)}{1 - \bar{K}}, \tag{5}$$

where

$$L = A^2(8 + 45\alpha^2 M^2) + 18\alpha^2(M^2\beta^2 + B) + \frac{\alpha^2}{\beta^2}(3D + \sigma_v^2).$$

We note that Theorem 1 establishes, in particular, the asymptotic boundary of the mean-square residues of estimation

$$\bar{L} = \sqrt{\frac{L + H/\delta}{K - \delta H}}.$$

This formula for the boundary $\bar{L}$ can be readily minimized with respect to δ as

$$\delta^\star = \frac{\sqrt{H^2 + KL} - H}{L}, \quad \bar{L}^\star = \frac{\sqrt{H^2 + KL} + H}{K}.$$

Conditions (**A**)–(**C**), (**E**) and (**F**) are standard for proving consistency of the estimates of the algorithms of stochastic approximation with input perturbations [2]. Mean-square stabilization of the estimates of algorithm (4) was proved earlier in [10] under more rigid constraints. Theorem 1 is proved in the Appendix.

5. EXAMPLE

We consider a rather simple application of the stochastic approximation algorithm with simultaneous trial input perturbation to minimization of the aforementioned nonstationary functional. Let it be required to estimate the coordinate of a point moving in the multidimensional space where the single possible measurement is the distance to it which is measured in noise. By virtue of Theorem 1, the estimates made by algorithm (4) will stabilize along the drift trajectory, provided that the norm of extremum drift is bounded.

We first consider a two-dimensional case with the model of drift obeying $\theta_n = \theta_{n-1} + \zeta$, where ζ is the vector $A = \|\zeta\| = 0.1$ uniformly distributed over a sphere. It is possible to consider a family of functions $F(x, w, n) = f(x, n) = \|x - \theta_n\|^2$ defining the square of distance to the desired point. These functions satisfy the conditions of Theorem 1. At numerical modeling, at each step the measurements were carried out with additional nonrandom noise $y_n = f(x_n, n) + v_n$, where the noise $v_n \in [-1, 1]$ was generated by the deterministic law $v_{2i} = 1 - (i \mod 3)$ for even steps and $v_{2i-1} = 1 - (i \mod 7)/3$ for odd ones. In the example under study the numerical parameters from the conditions of Theorem 1 are as follows: $A = 0.1$, $M = 2$, $B = 0$, $C = 0.04$, $D = 0.0004$, and $\mu = 2$. At that, according to Theorem 1 we obtain that $H = 0.1 + 2\alpha\beta$, $K = 4\alpha - 3(0.04 + 24\beta^2)\dfrac{\alpha^2}{\beta^2}$,
$L = 0.08 + 18\alpha^2(4\beta^2 + 0.1) + 1.0012\dfrac{\alpha^2}{\beta^2}$.

In the experiment, $\alpha = 1/36$ and $\beta = 1$ were selected. At that, $H = 0.106$; $K = 0.046$; $L = 0.159$. Having selected $\delta^\star = 0.192$, we obtain $\bar{L}^\star = 5.219$. The minimum point drifts as shown by the solid line in Fig. 1. The dashed line represents the drift of estimate obtained by algorithm (4). The initial value of the estimate $\hat{\theta}_0 = (25, 25)^{\mathrm{T}}$. The estimation error (norm of residual) and level of observation noise are shown in Fig. 2. These experimental results proved to be typical of diverse kinds of nonrandom bounded observation noise. As can be seen, the algorithm proposed by the

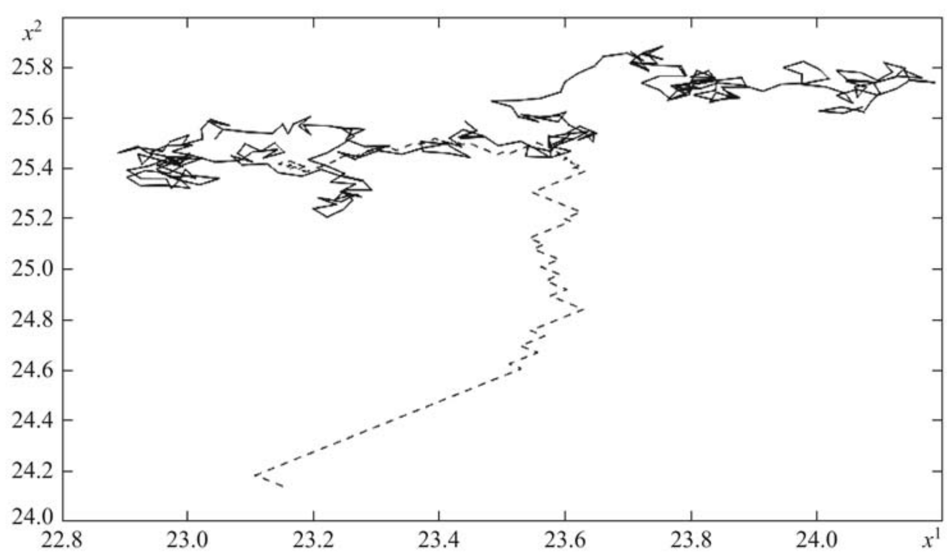

Fig. 1. Extremum θ_n (solid line) and its estimate (dashed line), method iterations 250–500.

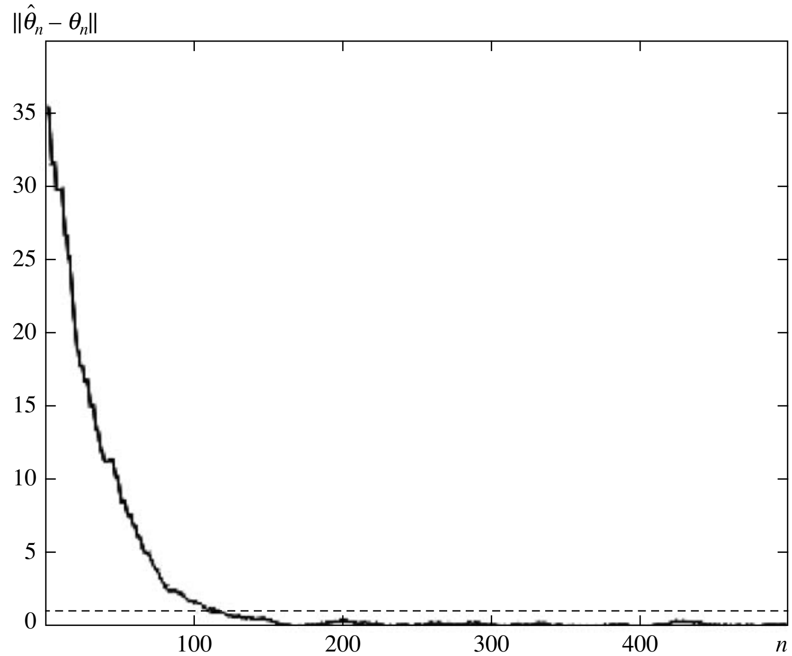

Fig. 2. Norm of estimation error and asymptotic boundary.

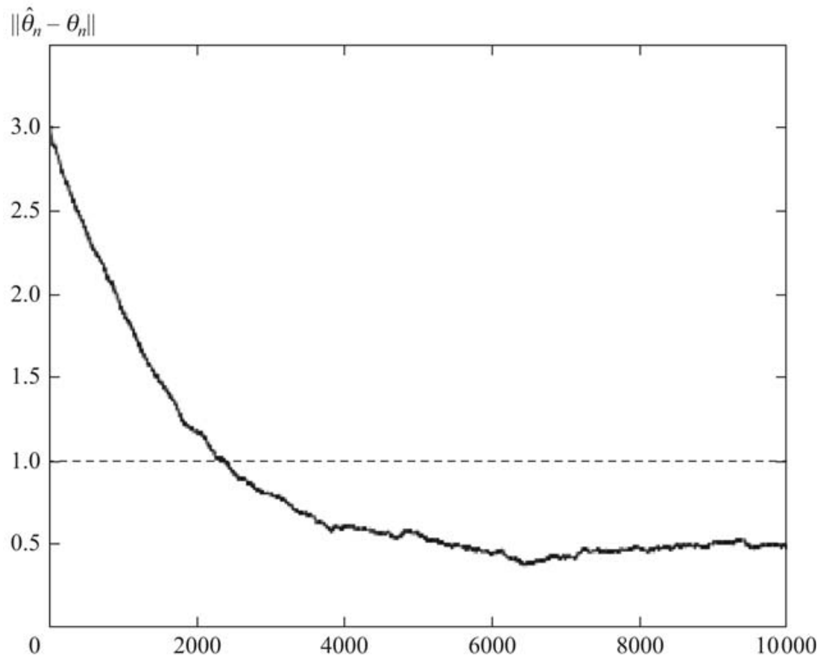

Fig. 3. Norm of error in the hundred-dimensional case.

present authors provides a level of residual much lower than the level of the observation noise, which cannot be attained by using the standard minimax methods under "unknown yet bounded" observation noise.

In the second example we considered the drift in a hundred-dimensional space. With such dimension, the standard algorithms based on approximation of the gradient vector use at each step 100 or 200 estimates, that is, the drift during one iteration is sufficiently appreciable. Figure 3 depicts variation of the level of estimate error for the drift rate in one iteration $A = 0.01$ and noise level $\sigma_v = 1$. In the typical cases, the mean-square estimation error at modeling is maintained at the level 0.5, which is much lower than the level of error accumulated in the methods based on the finite-difference approximations of the gradient.

6. CONCLUSIONS

The pseudogradient method with input randomization which is applicable in the case of measurements in noise of nonstatistical nature was used to estimate the minimum of nonstationary functional. It needs no measurements of the gradient and at each iteration requires only two measurements. The drift of extremum is regarded as bounded in norm at each step and, therefore, includes both random walks and motion in a certain direction. The error of estimation by this algorithm was proved to be bounded by a constant. Numerical modeling demonstrated that the error level is comparable with the level of drift which is assumed to be unknown.

The present authors plan to (i) establish an efficient asymptotic upper boundary for the sequence of estimates provided by the algorithm and (ii) strengthen the results obtained using the concepts of polynomial approximation of drift as formulated in [11]. This would substantially extend the conditions for stabilization of errors and enable one to reject the uniform drift boundedness in favor of the weaker condition of polynomial boundedness.

APPENDIX

Proof of Theorem 1. We denote $err_{n-1} = \hat\theta_{2n-2} - \theta_{2n-2}$, $drift_n = \theta_{2n} - \theta_{2n-2}$, $step_n = \dfrac{\alpha}{2\beta}(y_{2n} - y_{2n-1})\Delta_n$.

In virtue of algorithm (4) and the first on conditions (**D**) for the square of the norm of difference $\|\hat\theta_{2n} - \theta_{2n}\|$, we get the estimate

$$\begin{aligned}\|err_n\|^2 &\le \|err_{n-1}\|^2 + \|drift_n\|^2 + \|step_n\|^2 + 2\langle drift_n, step_n\rangle - 2\langle drift_n, err_{n-1}\rangle - 2\langle step_n, err_{n-1}\rangle \\ &\le \|err_{n-1}\|^2 + 4A^2 + \|step_n\|^2 + 2\langle step_n, drift_n\rangle - 2\langle drift_n, err_{n-1}\rangle - 2\langle err_{n-1}, step_n\rangle. \qquad \text{(A.1)}\end{aligned}$$

(1) In virtue of the observation model (2), we obtain for the last addend

$$\begin{aligned}&-\langle err_{n-1}, step_n\rangle \\ &= -\Big\langle err_{n-1}, \frac{\alpha}{2\beta}\Delta_n\Big(F\Big(\hat\theta_{2n-2} + \beta\Delta_n, w_{2n}, 2n\Big) - F\Big(\hat\theta_{2n-2} - \beta\Delta_n, w_{2n-1}, 2n-1\Big) + v_{2n} - v_{2n-1}\Big)\Big\rangle.\end{aligned}$$

The conditional expectation relative to the σ-algebra generated by the random variables $\theta_1, \ldots, \theta_{2n-2}$, $\hat\theta_1, \ldots, \hat\theta_{2n-2}$ is denoted by $E_n\{\cdot\}$. By applying $E_n\{\cdot\}$ to the last formula, using $E_n\{\Delta_n(v_{2n} - v_{2n-1})\} = 0$, adding and subtracting $F(\hat\theta_{2n-2}, w_{2n}, 2n)$ and $F(\hat\theta_{2n-2}, w_{2n-1}, 2n-1)$, we obtain

$$\begin{aligned}&E_n\{-\langle err_{n-1}, step_n\rangle\} \\ &= -\Big\langle err_{n-1}, \frac{\alpha}{2\beta}E_n\Big\{\Delta_n\Big(F\Big(\hat\theta_{2n-2} + \beta\Delta_n, w_{2n}, 2n\Big) - F\Big(\hat\theta_{2n-2}, w_{2n}, 2n\Big)\Big)\Big\}\Big\rangle \\ &-\Big\langle err_{n-1}, -\frac{\alpha}{2\beta}E_n\Big\{\Delta_n\Big(F\Big(\hat\theta_{2n-2} - \beta\Delta_n, w_{2n-1}, 2n-1\Big) - F\Big(\hat\theta_{2n-2}, w_{2n-1}, 2n-1\Big)\Big)\Big\}\Big\rangle \\ &-\Big\langle err_{n-1}, \frac{\alpha}{2\beta}E_n\Big\{\Delta_n\Big(F\Big(\hat\theta_{2n-2}, w_{2n}, 2n\Big) - F\Big(\hat\theta_{2n-2}, w_{2n-1}, 2n-1\Big)\Big)\Big\}\Big\rangle. \qquad \text{(A.2)}\end{aligned}$$

Let us consider the difference under the sign of $E_n\{\cdot\}$ in the first addend of (A.2). With due regard for the decomposition of $F(\hat{\theta}_{2n-2} + \beta\Delta_n, w_{2n}, 2n)$ by the Taylor formula, we derive successively that

$$E_n\left\{\Delta_n\left(F\left(\hat{\theta}_{2n-2} + \beta\Delta_n, w_{2n}, 2n\right) - F\left(\hat{\theta}_{2n-2}, w_{2n}, 2n\right)\right)\right\}$$
$$= E_n\left\{\Delta_n\left\langle\nabla F\left(\hat{\theta}_{2n-2} + \gamma_1\beta\Delta_n, w_{2n}, 2n\right), \beta\Delta_n\right\rangle\right\}$$
$$= E_n\left\{\Delta_n\left\langle\nabla F\left(\hat{\theta}_{2n-2}, w_{2n}, 2n\right), \beta\Delta_n\right\rangle\right\}$$
$$+ E_n\left\{\Delta_n\left\langle\nabla F\left(\hat{\theta}_{2n-2} + \gamma_1\beta\Delta_n, w_{2n}, 2n\right) - \nabla F\left(\hat{\theta}_{2n-2}, w_{2n}, 2n\right), \beta\Delta_n\right\rangle\right\},$$

for $\gamma_1 \in (0,1)$. As the result, application of (**A**)–(**C**) provides for the first addend in (A.2):

$$-\left\langle err_{n-1}, \frac{\alpha}{2\beta}E_n\left\{\Delta_n\left(F\left(\hat{\theta}_{2n-2} + \beta\Delta_n, w_{2n}, 2n\right) - F\left(\hat{\theta}_{2n-2}, w_{2n}, 2n\right)\right)\right\}\right\rangle$$
$$\leq \frac{\alpha}{2}\left(-\left\langle err_{n-1}, \nabla f\left(\hat{\theta}_{2n-2}, 2n\right)\right\rangle + M\beta\|err_{n-1}\|\right)$$
$$\leq \frac{1}{2}\left(-\alpha\mu\|err_{n-1}\|^2 + \alpha\beta M\|err_{n-1}\|\right). \tag{A.3}$$

A similar relation is obtained also for the second addend in (A.2). In virtue of independence of the trial perturbation Δ_n of w_{2n} and w_{2n-1}, we obtain for the third addend that

$$-\left\langle err_{n-1}, \frac{\alpha}{2\beta}E_n\left\{\Delta_n\left(F\left(\hat{\theta}_{2n-2}, w_{2n}, 2n\right) - F\left(\hat{\theta}_{2n-2}, w_{2n-1}, 2n-1\right)\right)\right\}\right\rangle = 0.$$

As the result, $-E_n\{\langle err_{n-1}, step_n\rangle\} \leq -\alpha\mu\|err_{n-1}\|^2 + \alpha\beta M\|err_{n-1}\|$.

(2) In virtue of condition (**D**), we obtain for the addend $2E_n\{\langle step_n, drift_n\rangle\}$:

$$2E_n\{\langle step_n, drift_n\rangle\} \leq E_n\{\|step_n\|^2\} + 4A^2.$$

(3) We consider $E_n\{\|step_n\|^2\}$. As in Item 1, by decomposing $y_{2n} - y_{2n-1}$ into several addends using the properties (**B**) and (**D**), we obtain the estimates

$$E_n\left\{\left\|F\left(\hat{\theta}_{2n-2}, w_{2n}, 2n\right) - F\left(\hat{\theta}_{2n-2}, w_{2n-1}, 2n-1\right)\right\|^2\right\} \leq C\|err_{n-1}\|^2 + D,$$
$$E_n\left\{\left\|F\left(\hat{\theta}_{2n-2} + \beta\Delta_n, w_{2n}, 2n\right) - F\left(\hat{\theta}_{2n-2}, w_{2n}, 2n\right)\right\|^2\right\}$$
$$+ E_n\left\{\left\|-F\left(\hat{\theta}_{2n-2} - \beta\Delta_n, w_{2n-1}, 2n-1\right) + F\left(\hat{\theta}_{2n-2}, w_{2n-1}, 2n-1\right)\right\|^2\right\}$$
$$\leq 3\beta^2\left(2(M^2\beta^2 + B) + M^2\left(\left\|\hat{\theta}_{2n-2} - \theta_{2n}\right\|^2 + \left\|\hat{\theta}_{2n-2} - \theta_{2n-1}\right\|^2\right)\right)$$
$$\leq 3\beta^2\left(M^2(2\beta^2 + 5A^2) + 2B + 2M^2\|err_{n-1}\|^2\right).$$

As the result,

$$E_n\{\|step_n\|^2\} \leq \frac{\alpha^2}{2\beta^2}\left(\|err_{n-1}\|^2(3C + 18M^2\beta^2) + 3D\right.$$
$$\left.+ 9\beta^2(M^2(2\beta^2 + 5A^2) + 2B) + E_n\{\|v_{2n} - v_{2n-1}\|^2\}\right).$$

Summation of the above estimates with regard for the form of K and H provides

$$E_n\{\|err_n\|^2\} \leq \|err_{n-1}\|^2(1-K) + 2\|err_{n-1}\|H$$
$$+ 8A^2 + 9\alpha^2(M^2(2\beta^2 + 5A^2) + 2B) + \frac{\alpha^2}{\beta^2}(3D + E_n\{\|v_{2n} - v_{2n-1}\|^2\}).$$

By passing to the unconditional expectation, we obtain in virtue of validity of the inequality $2aH \leq \delta Ha^2 + H/\delta \; \forall \delta > 0$ that

$$E\|err_n\|^2 \leq (1 - K + \delta H)E\|err_{n-1}\|^2 + L + H/\delta.$$

We select $0 < \delta < K/H$, which is possible by virtue of the condition of Theorem 1 for the parameters α and β of the algorithm, and derive (5) by an n-fold iteration of the last inequality.

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